A MINIMAL TYPE OF THE 2-ADIC WEIL REPRESENTATION

by

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ABSTRACT

A minimal type of the even 2-adic Weil representation is described. The Hecke algebra of this type is isomorphic to the classical affine Hecke algebra of type B_n. In this way, the Weil representation of a metaplectic group corresponds to the trivial representation of an orthogonal group in the local Shimura correspondence.

CONTENTS

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PREFACE

Let $\mathcal{W}$ be a nondegenerate symplectic space over a p-adic field, let $G = \widetilde{\mathrm{Sp}}(\mathcal{W})$ be a nontrivial two-fold central extension of the symplectic group $\underline{G} = \mathrm{Sp}(\mathcal{W})$ of type C_n, and let ω be the Weil representation of G. If p is odd, ω has a one-dimensional subspace upon which the inverse image of the Iwahori subgroup acts. This minimal type was used by Gan and Savin in [2] to establish an equivalence of categories between certain representations of G and certain other representations of orthogonal groups of type B_n. In this equivalence, the even Weil representation ω^e corresponds to the trivial representation of a split adjoint group of type B_n. In particular, the correspondence is realized as an isomorphism between the Hecke algebras of these types and the affine Hecke algebras of type B_n.

If $p = 2$, there are no Iwahori-fixed vectors of the Weil representation. However, for the two-fold cover $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ of $\mathrm{SL}_2(\mathbb{Q}_2)$, there is a one-dimensional type of the Weil representation for the first congruence subgroup of the Iwahori subgroup. The corresponding Hecke algebra was shown to be isomorphic to that of $\mathrm{PGL}_2(\mathbb{Q}_2)$ in [7]. The main purpose of this paper is to frame this result for $p = 2$ in the language of the Weil representation and to extend it to larger symplectic groups; that is, to find a minimal type of the Weil representation whose corresponding Hecke algebra is the affine Hecke algebra of type B_n.

The description of the appropriate open compact subgroup stems from the fact that in characteristic 2, a symmetric quadratic form is alternating. Hence, the finite orthogonal group $\mathrm{O}_{2n}(2)$ is a subgroup of the finite symplectic group $\mathrm{Sp}_{2n}(2)$. If B' is a Borel subgroup of $\mathrm{O}_{2n}(2)$ which sits in a Borel subgroup B of $\mathrm{Sp}_{2n}(2)$, then the inverse image $\underline{K}$ of B' (under the projection map $\mathbb{Z}_2 \to \mathbb{F}_2$) is a subgroup of index 2^n of the Iwahori subgroup, as pictured in the diagram below.

$$\begin{tikzcd}
\underline{K} \arrow[r] \arrow[d] & \underline{I} \arrow[r] \arrow[d] & \mathrm{Sp}_{2n}(\mathbb{Z}_2) \arrow[d] \\
B' \arrow[r] & B \arrow[r] & \mathrm{Sp}_{2n}(2)
\end{tikzcd}$$

In the SL_2 case, this subgroup $\underline{K}$ is exactly the subgroup from [7]. Let K be the inverse image in G of $\underline{K}$.

Chapter 1 of this paper is dedicated to fixing notation and summarizing previously known, but relevant, results. In Chapter 2, the $\widetilde{\mathrm{SL}}_2$ case is worked out in detail. Specifically, the one-dimensional subspace on which K acts is the span of the characteristic function of $\mathbb{Z}_2$. Since K is an index 2 subgroup of the Iwahori subgroup, the support of the associated Hecke algebra $\mathcal{H}$ is potentially much larger than that of the classical Iwahori-Hecke algebra. However, it is shown that $\mathcal{H}$ is supported exactly on those K-double cosets parametrized by the affine Weyl group. Using some general facts about Hecke algebras and finite-index subgroups, generators T_0 and T_1 for $\mathcal{H}$ are given which satisfy only the quadratic relations

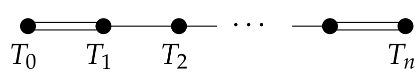
$$(T_0 - 2)(T_0 + 1) = 0 \quad \text{and} \quad (T_1 - 1)(T_1 + 1) = 0.$$

These are exactly the quadratic relations for the affine Hecke algebra of $\mathrm{PGL}_2(\mathbb{Q}_2)$.

In Chapter 3, the general symplectic case is described. In this setting, the one-dimensional K-type is the span of the characteristic function of the standard lattice, and the support of the associated Hecke algebra $\mathcal{H}$ is again reduced to those K-double cosets parametrized by the affine Weyl group. Drawing from, the $\widetilde{\mathrm{SL}}_2$ case, generators $T_0, \ldots, T_n$ for $\mathcal{H}$ are defined which satisfy the quadratic relations

$$(T_i - 2)(T_i + 1) = 0 \quad \text{and} \quad (T_n - 1)(T_n + 1) = 0.$$

The braid relations for these generators are given by the following Coxeter diagram.



The affine Hecke algebra A of type B_n has generators $t_0, \ldots, t_n$ which satisfy the quadratic relations $(t_i - 2)(t_i + 1) = 0$. The Coxeter diagram of type B_n has an involution given by $\tau t_1 \tau = t_0$, which satisfies $\tau^2 = 1$ and $\tau t_1 \tau t_1 = t_1 \tau t_1 \tau$. The isomorphism between $\mathcal{H}$ and A is given by

$$t_n \mapsto T_0, \quad \ldots \quad t_1 \mapsto T_{n-1} \quad \text{and} \quad \tau \mapsto T_n.$$

CHAPTER 1

PRELIMINARIES

In this chapter, notation is fixed and some relevant results are summarized. Sections 1.1 through 1.4 deal with symplectic groups and algebras; Section 1.5 summarizes the relationship between Hecke algebras and induced representations; Section 1.6 and 1.7 discuss additive characters and Fourier transforms over $\mathbb{Q}_p$; and Section 1.8 lays out the necessary Weil representation background.

1.1 Symplectic Vector Spaces

Let $\mathcal{W}$ be a $2n$-dimensional vector space over a field F, and let Q be a bilinear form on $\mathcal{W}$ which is

1. *skew-symmetric*: $Q(u, v) = -Q(v, u)$ for all $u, v \in \mathcal{W}$,
2. *nondegenerate*: if $Q(u, v) = 0$ for all v in $\mathcal{W}$, then $u = 0$.

Such a bilinear form is called a *symplectic form* and a vector space equipped with a symplectic form is called a *symplectic space*.

A bilinear form is called *totally isotropic* if $Q(v, v) = 0$ for all v in $\mathcal{W}$. As a remark, a totally isotropic bilinear form Q on $\mathcal{W}$ is always skew-symmetric, and a skew-symmetric form only fails to be totally isotropic if the characteristic of F is 2.

A subspace $\mathcal{X}$ of $\mathcal{W}$ is *isotropic* if Q is identically zero on $\mathcal{X}$. For a maximal isotropic subspace $\mathcal{X}$ of $\mathcal{W}$, there is a complementary subspace $\mathcal{Y}$ which is also a maximal isotropic subspace of $\mathcal{W}$, and each of $\mathcal{X}, \mathcal{Y}$ is a vector subspace of dimension n. A decomposition $\mathcal{W} = \mathcal{X} + \mathcal{Y}$ into maximal isotropic subspaces is called a *complete polarization* of $\mathcal{W}$.

Let $\mathcal{X} + \mathcal{Y}$ be a complete polarization of $\mathcal{W}$ and let $\{e_i\}$ be a basis of $\mathcal{X}$. There is a basis $\{f_i\}$ of $\mathcal{Y}$ such that $Q(e_i, f_j) = \delta_{ij}$. The resulting basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of $\mathcal{W}$ is called a *symplectic basis*. Under such a basis, elements of $\mathcal{W}$ may be considered as column vectors and the symplectic form Q may be expressed as

$$Q(u, v) = {}^{\intercal}u J v,$$

where ${}^{\intercal}u$ denotes the transpose of u and J is the $2n \times 2n$ matrix

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

1.2 Classical Symplectic Groups

Let $\mathcal{W}$ be a symplectic space of dimension $2n$ over a field F with symplectic form Q. The *symplectic group* $\mathrm{Sp}(\mathcal{W})$, or $\mathrm{Sp}_{2n}(F)$, is defined to be the group of linear automorphisms of $\mathcal{W}$ that preserve the symplectic form, i.e., those linear operators $T : \mathcal{W} \to \mathcal{W}$ such that for all u, v in $\mathcal{W}$,

$$Q(Tu, Tv) = Q(u, v).$$

Under a symplectic basis, operators $T : \mathcal{W} \to \mathcal{W}$ correspond to $2n \times 2n$ matrices and the symplectic group thus becomes a matrix group. Let x be the matrix corresponding to a transformation T. The property $Q(Tu, Tv) = Q(u, v)$ translates to ${}^{\intercal}u{}^{\intercal}xJxv = {}^{\intercal}uJv$, so the symplectic matrix group is precisely the set of those matrices x that satisfy the condition ${}^{\intercal}xJx = J$. In particular, if

$$x = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then the condition yields

$${}^{\intercal}ad - {}^{\intercal}cb = 1, \qquad {}^{\intercal}ac = {}^{\intercal}ca, \qquad \text{and} \qquad {}^{\intercal}bd = {}^{\intercal}db.$$

Moreover, if $n = 1$, then the only condition is $ad - bc = 1$, so $\mathrm{Sp}_2(F) = \mathrm{SL}_2(F)$.

1.3 Symplectic Lie Algebras

Let $\mathcal{W}$ be a $2n$-dimensional symplectic vector space over $\mathbb{C}$ with symplectic form Q. The *symplectic Lie algebra*, $\mathfrak{sp}(\mathcal{W})$ or $\mathfrak{sp}_{2n}(\mathbb{C})$, is defined to be the algebra of linear endomorphisms T of $\mathcal{W}$ satisfying

$$Q(Tu, v) + Q(u, Tv) = 0$$

for all u, v in $\mathcal{W}$. The set $\mathfrak{sp}(\mathcal{W})$ is a vector space under the usual addition and scalar multiplication of linear operators and an algebra under the Lie bracket

$$[T_1, T_2] = T_1T_2 - T_2T_1.$$

Under the symplectic basis, the Lie algebra $\mathfrak{sp}(\mathcal{W})$ becomes a subalgebra of the matrix algebra $\mathfrak{gl}_{2n}(\mathbb{C})$ of all $2n \times 2n$ matrices with complex entries. If X is the matrix corresponding to an endomorphism T of $\mathcal{W}$, then the condition $Q(Tu,v)+Q(u,Tv)=0$ translates to ${}^{\intercal}XJ+JX=0$. Writing

$$X=\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

this condition implies that $d=-{}^{\intercal}a$ and that b and c are symmetric. The symplectic Lie algebra is then realized as the algebra of matrices

$$\mathfrak{sp}(\mathcal{W})=\left\{\begin{bmatrix} a & b \\ c & -{}^{\intercal}a \end{bmatrix}\in\mathfrak{gl}_{2n}(\mathbb{C}) : b={}^{\intercal}b,\ c={}^{\intercal}c\right\}.$$

1.3.1 Cartan Decomposition

Let $\mathfrak{h}$ be the Cartan subalgebra consisting of diagonal matrices in $\mathfrak{sp}(\mathcal{W})$ and let $\mathfrak{h}^*=\mathrm{Hom}_{\mathbb{C}}(\mathfrak{h},\mathbb{C})$ be its linear dual. The Cartan subalgebra is n-dimensional and an arbitrary element H of $\mathfrak{h}$ is of the form

$$H=\begin{bmatrix} a & \\ & -a \end{bmatrix},$$

where a is a diagonal matrix with entries $a_1,\dots,a_n$. The dual basis $\{\lambda_1,\dots,\lambda_n\}$ of $\mathfrak{h}^*$ is defined by $\lambda_i(H)=a_i$.

Let E_{ij} be the $n\times n$ matrix with a 1 in the ijth position and 0 elsewhere. The one-dimensional subspaces of $\mathfrak{sp}(\mathcal{W})$ that lie outside of $\mathfrak{h}$ are spanned by the following matrices (where $i\neq j$):

$$\begin{bmatrix} E_{ji} & 0 \\ 0 & -E_{ij} \end{bmatrix},\quad \begin{bmatrix} 0 & E_{ii} \\ 0 & 0 \end{bmatrix},\quad \begin{bmatrix} 0 & 0 \\ E_{ii} & 0 \end{bmatrix},$$
$$\begin{bmatrix} 0 & E_{ij}+E_{ji} \\ 0 & 0 \end{bmatrix},\quad\text{and}\quad \begin{bmatrix} 0 & 0 \\ E_{ij}+E_{ji} & 0 \end{bmatrix}.$$

For the $H\in\mathfrak{h}$, the Lie bracket $[H,X]$ is given by

$$\begin{aligned}
(a_i-a_j)X &= (\lambda_i-\lambda_j)(H)X && \text{if } X \text{ is of the first type,}\\
2a_iX &= 2\lambda_i(H)X && \text{if } X \text{ is of the second type,}\\
-2a_iX &= -2\lambda_i(H)X && \text{if } X \text{ is of the third type,}\\
(a_i+a_j)X &= (\lambda_i+\lambda_j)(H)X && \text{if } X \text{ is of the fourth type,}\\
-(a_i+a_j)X &= -(\lambda_i+\lambda_j)(H)X && \text{if } X \text{ is of the fifth type.}
\end{aligned}$$

Therefore, the roots in $\mathfrak{h}^*$ are

$$\Phi=\left\{\pm\lambda_i\pm\lambda_j : i\neq j\right\}\cup\left\{\pm 2\lambda_i\right\}.$$

The roots in the first set are called *short roots* and those in the second set are called *long roots*. If the elementary matrix corresponding to a root α is denoted by X_α, then the root space of α is $\mathbb{C}X_\alpha$, and the Cartan decomposition is

$$\mathfrak{sp}(\mathcal{W})=\mathfrak{h}\oplus\sum_{\alpha\in\Phi}\mathbb{C}X_\alpha.$$

1.3.2 Roots

Consider the real vector space $\mathfrak{h}_{\mathbb{R}}^*$, defined to be the $\mathbb{R}$-span of $\{\lambda_i\}$. This vector space is a Euclidean space under the usual dot product, denoted $(\ ,\ )$, which is symmetric and positive definite. This inner product is the same as the one which comes from the Killing form on $\mathfrak{h}$ (see [9]). For convenience, if $\mu, \lambda \in \mathfrak{h}_{\mathbb{R}}^*$, write

$$\langle \mu, \lambda \rangle = \frac{2(\mu, \lambda)}{(\lambda, \lambda)}.$$

For each $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$, define the reflection s_λ on $\mathfrak{h}_{\mathbb{R}}^*$ by

$$s_\lambda(\mu) = \mu - \langle \mu, \lambda \rangle \lambda.$$

Proposition 1.1 *The set of roots Φ forms a root system in $\mathfrak{h}_{\mathbb{R}}^*$, and $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ forms a set of simple roots in Φ, where $\alpha_i = \lambda_i - \lambda_{i+1}$, for $i = 1, \ldots, n-1$, and $\alpha_n = 2\lambda_n$. The corresponding set of positive roots is $\Phi^+ = \{\lambda_i \pm \lambda_j : i < j\} \cup \{2\lambda_i\}$.*

Proof: First, Φ clearly generates $\mathfrak{h}_{\mathbb{R}}^*$ since each $2\lambda_i$ is in Φ. Second, a quick check ensures that α and $k\alpha$ are in Φ if and only if $k = \pm 1$. Third, if $\alpha, \beta \in \Phi$, then $\langle \alpha, \beta \rangle$ is an integer since the dot product of any short root with itself is 2, the dot product of any long root with itself is 4, and the dot product of a long root with any other root is a multiple of 2. Of particular interest are the *Cartan integers*, $c_{ij} = \langle \alpha_i, \alpha_j \rangle$ for $\alpha_i, \alpha_j \in \Pi$. The matrix (c_{ij}) of Cartan integers is the following.

$$\begin{bmatrix} 2 & -1 & & & & & & & \\ -1 & 2 & -1 & & & & & & \\ & -1 & 2 & -1 & & & & & \\ & & -1 & & & & & & \\ & & & & \ddots & & & & \\ & & & & & & -1 & & \\ & & & & & -1 & 2 & -1 & \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -2 & 2 \end{bmatrix}$$

Fourth, it must be verified that Φ is preserved by any reflection s_α for $\alpha \in \Phi$. If $\alpha = \pm(\lambda_i - \lambda_j)$, then s_α just interchanges λ_i and λ_j; if $\alpha = \pm(\lambda_i + \lambda_j)$, then s_α interchanges λ_i and $-\lambda_j$; and if $\alpha = \pm 2\lambda_i$, then s_α takes λ_i to $-\lambda_i$. Clearly, the set Φ is invariant under these operations. Therefore, Φ is indeed a root system.

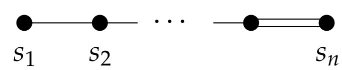
It remains to be seen that the set Π is a system of simple roots; that is, Π must be a basis for $\mathfrak{h}_{\mathbb{R}}^*$ such that each $\alpha \in \Phi$ may be expressed as an integer combination of elements in Π

with either all positive coefficients (called *positive roots*) or all negative coefficients (called *negative roots*). The following formulas hold for $i < j$.

$$
\begin{aligned}
2\lambda_i &= 2\alpha_i + \cdots + 2\alpha_{n-1} + \alpha_n, \\
\lambda_i - \lambda_j &= \alpha_i + \cdots + \alpha_{j-1}, \\
\lambda_i + \lambda_j &= (\lambda_i - \lambda_j) + 2\lambda_j.
\end{aligned}
$$

These are the positive roots. To obtain the negative roots, one simply multiplies each expression by -1. Since the dimension of $\mathfrak{h}_{\mathbb{R}}^*$ is $n = |\Pi|$, the linear independence of Π is automatic. ■

For $i = 1, \ldots, n$, define s_i to be the reflection s_{α_i}. The Weyl group W of $\mathfrak{sp}(\mathcal{W})$ is the finite group of reflections s_α for α in Φ. It is a Coxeter group and is generated by the set of simple reflections $\{s_1, \ldots, s_n\}$. The braid relations for the generators of W are given by the following Coxeter diagram (see [3]).



1.3.3 Affine Roots

The set of affine roots is the set

$$\Phi^{\text{aff}} = \{\alpha + n : \alpha \in \Phi, n \in \mathbb{Z}\}$$

of affine functionals on $\mathfrak{h}$ which act by $(\alpha + n)(H) = \alpha(H) + n$. Let α_* be the highest root in Φ given by

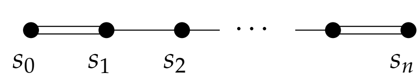
$$\alpha_* = 2\lambda_1 = 2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n,$$

and define the affine root α_0 by

$$\alpha_0 = 1 - \alpha_*.$$

For each affine root γ, the affine reflection s_γ is the reflection in $\mathfrak{h}$ across the hyperplane $\{H : s_\gamma(H) = 0\}$. For instance, if α is a root, the affine reflection s_α is the reflection defined in the previous section. Define s_0 to be the affine reflection $s_0 = s_{\alpha_0}$.

The group W^{aff} of affine reflections is called the affine Weyl group; it is a Coxeter group and is generated by the simple affine reflections $\{s_0, s_1, \ldots, s_n\}$. The braid relations for these generators of W^{aff} are given by the following Coxeter diagram (see [3]).



1.3.4 Chevalley Basis

The Chevalley basis for $\mathfrak{sp}(\mathcal{W})$, considered as a matrix algebra, will be given explicitly. For each $\alpha \in \Phi^+$, define

$$H_\alpha = [X_\alpha, X_{-\alpha}],$$

which is an element of $\mathfrak{h}$. This fact can be verified for each type of X_α, bearing in mind that $X_{-\alpha}$ is just the transpose of X_α. Indeed,

$$\begin{aligned}
\text{if } X_\alpha &= \begin{bmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{bmatrix}, & \text{then} \quad H_\alpha &= \begin{bmatrix} E_{ii} - E_{jj} & 0 \\ 0 & -E_{ii} + E_{jj} \end{bmatrix}; \\
\text{if } X_\alpha &= \begin{bmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{bmatrix}, & \text{then} \quad H_\alpha &= \begin{bmatrix} E_{ii} + E_{jj} & 0 \\ 0 & -E_{ii} - E_{jj} \end{bmatrix}; \\
\text{if } X_\alpha &= \begin{bmatrix} 0 & E_{ii} \\ 0 & 0 \end{bmatrix}, & \text{then} \quad H_\alpha &= \begin{bmatrix} E_{ii} & 0 \\ 0 & -E_{ii} \end{bmatrix}.
\end{aligned}$$

Lemma 1.2 *The following properties hold for all $\alpha, \beta \in \Phi$.*

1. $[H_\alpha, H_\beta] = 0$.
2. $[H_\alpha, X_\beta] = \langle \beta, \alpha \rangle X_\beta$.
3. $[X_\alpha, X_\beta] = \begin{cases} H_\alpha & \text{if } \beta = -\alpha, \\ c(\alpha, \beta) X_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi, \\ 0 & \text{otherwise.} \end{cases}$

   where $c(\alpha, \beta)$ is either ± 1 or ± 2.

Proof: The first property holds since $\mathfrak{h}$ is abelian. To see why the second property is true, recall that the bilinear form $(\ ,\ )$ is the same as the one which arises from the Killing form. More specifically, if κ denotes the Killing form, then, since κ is nondegenerate, for each $\lambda \in \mathfrak{h}^*$, there exists $H'_\lambda \in \mathfrak{h}$ such that $\kappa(H'_\lambda, H) = \lambda(H)$. It may be verified directly (see [9]) that the Killing form is connected to the bilinear form on $\mathfrak{h}^*$ by

$$(\lambda, \mu) = \kappa(H'_\lambda, H'_\mu),$$

and that for $\alpha \in \Phi$,

$$H_\alpha = \frac{2}{(\alpha, \alpha)} H'_\alpha.$$

Therefore,

$$[H_\alpha, X_\beta] = \alpha(H_\beta) X_\alpha = \kappa(H'_\alpha, H_\beta) X_\alpha = \frac{2}{(\beta, \beta)} \kappa(H'_\alpha, H'_\beta) X_\alpha = \langle \alpha, \beta \rangle X_\alpha.$$

The third property follows from the general fact that if α, β are roots of a Lie algebra $\mathfrak{g}$ with root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, which is nonzero if and only if $\alpha + \beta$ is a root.

The exact value of $c(\alpha,\beta)$ is simple to compute, but it is not very useful to write down all of the cases. It will suffice to note that

$$c(\alpha,\beta) = \begin{cases} \pm 2 & \text{if } (\alpha,\beta) = 0, \\ \pm 1 & \text{otherwise,} \end{cases}$$

which coincides with the usual notion that $c(\alpha,\beta) = \pm(p+1)$ where $\beta - p\alpha, \ldots, \beta + q\alpha$ is the α-string of roots through β. For more details, see [1]. ■

Proposition 1.3 *The set* $S = \{X_\alpha : \alpha \in \Phi\} \cup \{H_\alpha : \alpha \in \Pi\}$ *forms a Chevalley basis for* $\mathfrak{sp}(\mathcal{W})$.

Proof: The previous lemma shows that the elements of S satisfy the necessary properties to be a Chevalley basis. All that remains to be seen is that S is indeed a basis for $\mathfrak{sp}(\mathcal{W})$. The Cartan decomposition

$$\mathfrak{sp}(\mathcal{W}) = \mathfrak{h} \oplus \sum_{\alpha\in\Phi} \mathbb{C}X_\alpha$$

reduces the problem to showing that $\{H_\alpha : \alpha \in \Pi\}$ spans $\mathfrak{h}$, which follows directly from the fact that

$$\begin{bmatrix} E_{ii} & \\ & -E_{ii} \end{bmatrix} = H_{\alpha_i} + H_{\alpha_{i+1}} + \cdots + H_{\alpha_{n-1}} + \frac{1}{2}H_{\alpha_n}.$$

■

Corollary 1.4 *For each root* $\alpha \in \Phi$*, the* $\mathbb{C}$*-span of* $\{X_\alpha, X_{-\alpha}, H_\alpha\}$ *is a subalgebra of* $\mathfrak{sp}(\mathcal{W})$ *which is isomorphic to* $\mathfrak{sl}_2(\mathbb{C})$.

Proof: The isomorphism is given by

$$X_\alpha \leftrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad X_{-\alpha} \leftrightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad H_\alpha \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

■

1.3.5 Universal Enveloping Algebra

Let $\mathcal{U}$ be the universal enveloping algebra of $\mathfrak{sp}(\mathcal{W})$. The Poincaré-Birkhoff-Witt Theorem implies that $\mathfrak{sp}(\mathcal{W})$ can be embedded into $\mathcal{U}$ and that if $\{Y_1, \ldots, Y_r\}$ is an ordering of the basis $\{X_\alpha : \alpha \in \Phi\} \cup \{H_\alpha : \alpha \in \Pi\}$ of $\mathfrak{sp}(\mathcal{W})$, then the set of monomials $Y_1^{m_1} \cdots Y_r^{m_r}$ forms a basis for $\mathcal{U}$ (see [4]).

1.4 Symplectic Chevalley Groups

Following [9], the classical symplectic group $\mathrm{Sp}_{2n}(F)$ will be constructed from the symplectic Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C})$. As in the previous section, let $\mathcal{W}$ be a symplectic vector space over $\mathbb{C}$, let $\mathcal{U}$ be the universal enveloping algebra of $\mathfrak{sp}(\mathcal{W})$, and let $\mathcal{U}_{\mathbb{Z}}$ be the subalgebra of $\mathcal{U}$ generated by the divided powers $X_\alpha^m/m!$. Under the natural representation of $\mathfrak{sp}(\mathcal{W})$, and hence of $\mathcal{U}$, on $\mathcal{W}$, the elements of $\mathcal{U}$ may be viewed as members of the matrix algebra. In this setting, each $X_\alpha^2 = 0$, so $\mathcal{U}_{\mathbb{Z}}$ is generated by 1 and $\{X_\alpha : \alpha \in \Phi\}$. Therefore, since $\mathcal{W}$ is the natural representation, the standard lattice

$$L = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n \oplus \mathbb{Z}f_1 \oplus \cdots \oplus \mathbb{Z}f_n$$

is the $\mathcal{U}_{\mathbb{Z}}$-invariant lattice guaranteed by Corollary 1 of Theorem 2 in [9].

Fix a field F; for an element t in F and a root α in Φ, one obtains the natural action of

$$x_\alpha(t) = \exp(tX_\alpha) = \sum_{n=0}^{\infty} t^n \frac{X_\alpha^n}{n!} = 1 + tX_\alpha$$

on $\mathcal{W}_F = L \otimes_{\mathbb{Z}} F$, which is the F-span of $e_1, \ldots, e_n, f_1, \ldots, f_n$. In other words, $x_\alpha(t)$ may be interpreted as the actual matrix $1 + tX_\alpha$ with entries in F.

For the data $\big(\mathfrak{sp}(\mathcal{W}), \mathcal{W}, F\big)$, one obtains a Chevalley group G, generated by

$$\{x_\alpha(t) : t \in F, \alpha \in \Phi\},$$

and, for each α, the root group

$$\mathfrak{X}_\alpha = \{x_\alpha(t) : t \in F\}.$$

The following elements play an important role in the theory of Chevalley groups. For $t \in F^\times$, define

$$\begin{aligned} w_\alpha(t) &= x_\alpha(t)x_{-\alpha}(-1/t)x_\alpha(t), \\ h_\alpha(t) &= w_\alpha(t)w_\alpha(-1). \end{aligned}$$

In addition, define the elements $w_1, \ldots, w_n$ of G by

$$w_i = w_{\alpha_i}(1),$$

where $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ is the set of simple roots. Then $\{w_1, \ldots, w_n\}$ is a set of representatives of generators of the Weyl group W in G (see [9] or [1]).

Suppose now that F is a nonarchimedian local field with uniformizer ϖ and ring of integers $\mathcal{O}$. For any affine root $\gamma = \alpha + n$, where α is a root and n is an integer, define the element

$$x_\gamma(t) = x_\alpha(\varpi^n t),$$

define G_γ to be the subgroup of G generated by $\{x_\gamma(t), x_{-\gamma}(t) : t \in F\}$, and define K_γ to be the subgroup generated by $\{x_\gamma(t), x_{-\gamma}(t) : t \in \mathcal{O}\}$. Consider the map $e_\gamma : \mathrm{SL}_2(F) \to G$ given by

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mapsto x_\gamma(t), \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \mapsto x_{-\gamma}(t).$$

Under e_γ, the image of $\mathrm{SL}_2(F)$ is G_γ and the image of $\mathrm{SL}_2(\mathcal{O})$ is K_γ.

In addition, for the affine root $\gamma = \alpha + n$, define the element

$$w_\gamma(t) = w_\alpha(\varpi^n t)$$

of G_γ. For the simple affine root $\alpha_0 = 1 - \alpha_*$ from Section 1.3.3, define the element w_0 of G to be

$$w_0 = w_{\alpha_0}(1) = w_{\alpha_*}(\varpi^{-1}).$$

The set $\{w_0, w_1, \ldots, w_n\}$ forms a set of representatives in G of $\{s_0, s_1, \ldots, s_n\}$, the simple reflections that generate the affine Weyl group (see [5]).

Proposition 1.5 *The Chevalley group G is isomorphic to the symplectic group* $\mathrm{Sp}(\mathcal{W}_F)$.

Proof: The classical symplectic group $\mathrm{Sp}(\mathcal{W}_F)$ and the Chevalley group G can both be thought of as matrix groups. Row and column reduction (relative to the symplectic form!) of a matrix in $\mathrm{Sp}(\mathcal{W}_F)$ is the same as multiplication on the right or left by the matrices $x_\alpha(t)$. One can row/column reduce a matrix in $\mathrm{Sp}(\mathcal{W}_F)$ to a representative w of an element of the Weyl group. Viewing G as a matrix group, the elements $w_\alpha(1)$ correspond exactly to those "signed" transposition matrices that represent elements of the Weyl group W of $\mathrm{Sp}(\mathcal{W}_F)$. Therefore, w can be expressed as a product of $w_\alpha(1)$'s, and hence, every matrix in $\mathrm{Sp}(\mathcal{W}_F)$ can be written as the product of $x_\alpha(t)$'s. ■

1.4.1 Generators and Relations

A summary of the results on generators and relations in [9] for the symplectic Chevalley group G is the following. First, G, which is generated by the $x_\alpha(t)$, satisfies the following relations: for any $\alpha, \beta \in \Phi$ and $t, u \in F$ (nonzero where necessary), (R1) $x_\alpha(t)x_\alpha(u) = x_\alpha(t+u)$;

(R2) $\big(x_\alpha(t), x_\beta(u)\big) = \prod x_{i\alpha+j\beta}(c_{ij}t^iu^j)$, for $\alpha+\beta \neq 0$;

(R3) $w_\alpha(t)x_\alpha(u)w_\alpha(-t) = x_{-\alpha}(-t^{-2}u)$;

(R4) $w_\alpha(1)x_\beta(t)w_\alpha(-1) = x_{s_\alpha(\beta)}(ct)$;

(R5) $w_\alpha(1)h_\beta(t)w_\alpha(-1) = h_{s_\alpha(\beta)}(t)$;

(R6) $h_\alpha(t)x_\beta(u)h_\alpha(t)^{-1} = x_\beta\big(t^{\langle\beta,\alpha\rangle}u\big)$;

(R7) $h_\alpha(t)h_\alpha(u) = h_\alpha(tu)$.

In (R2), the product is taken over the roots that are positive linear combinations of α and β, and $c_{ij} = c_{ij}(\alpha,\beta)$ is ± 1 if the angle between the roots is $3\pi/4$ and ± 2 if the angle is $\pi/2$. In (R4), $c = c(\alpha,\beta) = c(\alpha,-\beta) = \pm 1$.

Second, relations (R1) through (R6) are consequences of (R1) and (R2) if $G = \mathrm{Sp}_{2n}(F)$ for $n \geq 2$ and of (R1) and (R3) if $G = \mathrm{SL}_2(F)$. Hence, these two relations along with relation (R7) form a complete set of relations for G.

1.4.2 Central Extensions

A group G' is a *central extension* of a group G if there is a surjective homomorphism from G' to G whose kernel lies in the center of G'. A central extension E of G is *universal* if it is a central extension of any other central extension. By Theorem 10 of [9], a universal central extension E of the symplectic Chevalley group $\mathrm{Sp}_{2n}(F)$ exists and is the group defined abstractly using only relations (R1) and (R2). In the $\mathrm{SL}_2(F)$ case, the universal central extension is abstractly defined using only relations (R1) and (R3). Hence, in all cases, the relations (R1) – (R6) can be lifted to any central extension of G.

Let G' be any central extension of G. The preceding paragraph implies that the elements $x_\alpha(t)$ lift *uniquely* to elements $x'_\alpha(t)$ in G'. Therefore, $w_\alpha(t)$ and $h_\alpha(t)$ lift *canonically* to $w'_\alpha(t)$ and $h'_\alpha(t)$ in G' via the formulas

$$w'_\alpha(t) = x'_\alpha(t)x'_{-\alpha}(-1/t)x'_\alpha(t),$$
$$h'_\alpha(t) = w'_\alpha(t)w'_\alpha(-1).$$

The relations (R1) – (R6) hold in G', but the relation (R7) does not. Therefore, the symbol (t,u) given by $h'_\alpha(t)h'_\alpha(u) = (t,u)h'_\alpha(tu)$ provides the necessary information about this central extension. If G' is the universal central extension, then (t,u) is called the *Steinberg symbol*.

1.5 Induced Representations and Hecke Algebras

Let H be a finite index subgroup of an arbitrary group G, and suppose that G admits a left Haar measure normalized to give $\mathrm{vol}(H) = 1$. The space of complex-valued functions on G will be denoted $C(G)$. Fix a complex representation (π, V) of G and a character $\chi : H \to \mathbb{C}^\times$ of H. The objects of interest are

1. the induced representation $(\sigma, U) = \mathrm{ind}_H^G \chi$, where σ acts by right translation on the vector space

$$U = \{\phi \in C(G) : \phi(hg) = \chi(h)\phi(g) \text{ for } h \in H, g \in G\}\,;$$

2. the subspace $V^{H,\bar{\chi}}$ of V given by

$$V^{H,\bar{\chi}} = \big\{v \in V : \pi(h)v = \bar{\chi}(h)v \text{ for } h \in H\big\};$$

3. the Hecke algebra $\mathcal{H} = \mathcal{H}(G/\!/H; \chi)$ defined by

$$\mathcal{H} = \big\{f \in C(G) : f(h_1 g h_2) = \chi(h_1)f(g)\chi(h_2) \text{ for } h_1, h_2 \in H, g \in G\big\}.$$

Proposition 1.6 (Frobenius Reciprocity) *The spaces*

$$\begin{aligned}
\mathrm{Hom}_H(\pi, \chi) &= \big\{S : V \to \mathbb{C} \;:\; S\big(\pi(h)v\big) = \chi(h)S(v) \textit{ for } h \in H, v \in V\big\} \\
\mathrm{Hom}_G(\pi, \sigma) &= \big\{T : V \to U \;:\; T\big(\pi(g)v\big) = \sigma(g)T(v) \textit{ for } g \in G, v \in V\big\}
\end{aligned}$$

are isomorphic.

Proof: The map $S \mapsto T$ is given by $T(v)(g) = S(\pi(g)v)$. Indeed,

$$T(v)(hg) = S\big(\pi(hg)v\big) = \chi(h)S\big(\pi(g)v\big) = \chi(h)T(v)(g)$$

implies that $T(v) \in U$, while

$$T\big(\pi(g_2)v\big)(g_1) = S\big(\pi(g_1 g_2)v\big) = T(v)(g_1 g_2) = \big(\sigma(g_2)T(v)\big)(g_1)$$

implies that $T \in \mathrm{Hom}_G(\pi, \sigma)$. The map $T \mapsto S$ is given by $S(v) = T(v)(1)$. Clearly, $S(v) \in \mathbb{C}$ and

$$S\big(\pi(h)v\big) = T\big(\pi(h)v\big)(1) = T(v)(h) = \chi(h)T(v)(1) = \chi(h)S(v)$$

implies that $S \in \mathrm{Hom}_H(\pi, \chi)$. In addition, the composition of these maps is the identity. ■

Proposition 1.7 *The subspace $V^{H,\bar{\chi}}$ is an $\mathcal{H}$-module under the action*

$$\pi(f)v = \int_G f(g)\big(\pi(g)v\big)dg.$$

Proof: The first thing is to verify that $\pi(f)v$ is in $V^{H,\bar{\chi}}$ for $v \in V^{H,\bar{\chi}}$, $h \in H$, and $f \in \mathcal{H}$. This is done by a simple change of variables.

$$\pi(h)\big(\pi(f)v\big) = \int_G f(g)\big(\pi(h)\pi(g)v\big)dg = \int_G f(h^{-1}g)\big(\pi(g)v\big)dg = \bar{\chi}(h)\big(\pi(f)v\big).$$

It remains to check that the action is compatible with the structure of $\mathcal{H}$. It is easy to see that π is linear on $\mathcal{H}$, so it must only be verified that $\pi(f_1 * f_2) = \pi(f_1)\pi(f_2)$ for $f_1, f_2 \in \mathcal{H}$. The action π of $\mathcal{H}$ on $V^{H,\chi}$ may be expressed as a finite sum,

$$\begin{aligned}
\pi(f)v &= \sum_{x\in G/H} \int_{xH} f(g)\big(\pi(g)v\big)dg \\
&= \sum_{x\in G/H} \int_{H} f(xh)\big(\pi(xh)v\big)dh \\
&= \sum_{x\in G/H} f(x)\big(\pi(x)v\big),
\end{aligned}$$

and the convolution may also be expressed as a finite sum,

$$\begin{aligned}
(f_1 * f_2)(x) &= \int_G f_1(g)f_2(g^{-1}x)dg \\
&= \sum_{y\in G/H} \int_{yH} f_1(g)f_2(g^{-1}x)dg \\
&= \sum_{y\in G/H} \int_{H} f_1(yh)f_2(h^{-1}y^{-1}x)dh \\
&= \sum_{y\in G/H} f_1(y)f_2(y^{-1}x).
\end{aligned}$$

Under these formulations,

$$\begin{aligned}
\pi(f_1 * f_2)v &= \sum_{x\in G/H} (f_1 * f_2)(x)\pi(x)v \\
&= \sum_{x,y\in G/H} f_1(y)f_2(y^{-1}x)\pi(x)v \\
&= \sum_{x,y\in G/H} f_1(y)f_2(x)\pi(yx)v \\
&= \sum_{y\in G/H} f_1(y)\pi(y)\big(\pi(f_2)v\big) \\
&= \pi(f_1)\big(\pi(f_2)v\big).
\end{aligned}$$

■

Proposition 1.8 *The Hecke algebra $\mathcal{H}$ is isomorphic to the endomorphism ring of U. Moreover, elements of $\mathcal{H}$ not supported on H act on U as trace-zero endomorphisms.*

Proof. Define a map $\mathcal{H} \to \operatorname{End}(U)$ by $f \mapsto T$ with $T(\phi) = f * \phi$. $T(\phi)$ is indeed an element of U since

$$T(\phi)(hg) = \int_G f(x)\phi(x^{-1}hg)dx = \int_G f(yh)\phi(y^{-1}g)dy = \chi(h)T(\phi)(g).$$

Define a map $\operatorname{End}(U) \to \mathcal{H}$ by $T \mapsto f$ with $f(g) = T(\dot{\chi})(g)$, where $\dot{\chi}$ is the extension of χ by zero to all of G. This f is an element of $\mathcal{H}$, since $\dot{\chi}$ being an element of $U^{H,\chi}$ implies that

$$f(h_1gh_2) = T(\dot{\chi})(h_1gh_2) = \chi(h_1)\sigma(h_2)\big(T(\dot{\chi})(g)\big) = \chi(h_1)f(g)\chi(h_2).$$

The composition $\mathcal{H} \to \operatorname{End}(U) \to \mathcal{H}$ of these two maps is the identity, since

$$T(\dot{\chi})(g) = \int_G f(x)\dot{\chi}(x^{-1}g)dx = \int_H f(x^{-1}g)\chi(x)dg = f(g).$$

Suppose that $f \in \mathcal{H}$ is not supported on H. Since the functions $\operatorname{char}(Hx)$, for $x \in H\backslash G$, form a basis of U, to check that the trace of f is zero, it suffices to see that $(f * \phi)(x) = 0$ for $\phi \in U$ with $\operatorname{supp}(\phi) = Hx \neq H$. Indeed, for such a ϕ,

$$(f * \phi)(x) = \int_G f(g)\phi(g^{-1}x)dg = \begin{cases} \int_G 0 \cdot \phi(g^{-1}x)dg = 0 & \text{if } g \in H, \\ \int_G f(g) \cdot 0 \, dg = 0 & \text{if } g \notin H, \end{cases}$$

so the proposition is proved. ■

These two $\mathcal{H}$-modules are compatible in the following way. Let $V_1, \ldots, V_n$ be the irreducible representations of G such that $V_i^{H,\bar{\chi}}$ is nontrivial. For $v_i^* \in V_i^*$ and $v_i \in V_i$, the matrix coefficients $v_i^*(\pi_i(g^{-1})v_i)$ give a part $\bigoplus V_i \otimes V_i^*$ of the decomposition of $C(G)$; hence, the induced representation, since $U^{H,\bar{\chi}}$ is nontrivial, decomposes as

$$U = \bigoplus_{i=1}^{n} V_i^{H,\bar{\chi}} \otimes V_i^*.$$

In particular, if $\phi(g) = v_i^*(\pi_i(g^{-1})v_i)$, then

$$(f * \phi)(x) = \int_G f(g)v_i^*\big(\pi_i(x^{-1}g)v_i\big)dg = v_i^*\big(\pi_i(x^{-1})(\pi_i(f)v_i)\big) = \pi_i(f)(v_i \otimes v_i^*).$$

1.6 Characters of $\mathbb{Q}_p$

Let ψ be a nontrivial smooth additive character of $\mathbb{Q}_p$. As ψ is locally constant, there exists a smallest integer c , called the *conductor* of ψ, such that ψ is trivial on $p^c\mathbb{Z}_p$. The proof of the following proposition can be found in [10].

Proposition 1.9 *Fix a nontrivial smooth character ψ of $\mathbb{Q}_p$. Every smooth character of $\mathbb{Q}_p$ is of the form $x \mapsto \psi(ax)$ for some a in $\mathbb{Q}_p$.*

There is a natural projection from $\mathbb{Q}_p$ to $\mathbb{Q}_p/\mathbb{Z}_p$, a canonical embedding of $\mathbb{Q}_p/\mathbb{Z}_p$ into the p-torsion of $\mathbb{Q}/\mathbb{Z}$, and a nearly canonical embedding of $\mathbb{Q}/\mathbb{Z}$ into $\mathbb{C}^\times$ given by $x \mapsto e^{2\pi i x}$. The composition of these maps

$$\mathbb{Q}_p \twoheadrightarrow \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}^\times$$

defines an additive character $\psi_1 : \mathbb{Q}_p \to \mathbb{C}^\times$ of conductor $c = 0$.

For a in $\mathbb{Q}_p^\times$, define the character $\psi_a : \mathbb{Q}_p \to \mathbb{C}^\times$ by

$$\psi_a(x) = \psi_1(ax),$$

which has conductor $c = -\operatorname{val}(a)$. According to Proposition 1.9, these ψ_a account for all nontrivial smooth additive characters of $\mathbb{Q}_p$.

1.7 Fourier Transform

Let $S(\mathbb{Q}_p)$ denote the set of Schwarz functions on $\mathbb{Q}_p$, i.e., the set of smooth, compactly supported, complex-valued functions on $\mathbb{Q}_p$, and fix ψ to be an additive character of $\mathbb{Q}_p$ of conductor c. The Fourier transform (with respect to ψ) on $S(\mathbb{Q}_p)$ is given by $\phi \mapsto \widehat{\phi}$ where

$$\widehat{\phi}(y) = \int_{\mathbb{Q}_p} \psi(2uy)\phi(u)du.$$

For a given ψ, the Haar measure on $\mathbb{Q}_p$ will be normalized so that $\widehat{\widehat{\phi}}(y) = \phi(-y)$. The appearance of a 2 in the Fourier transform affects the role of the conductor of ψ for $p = 2$, so it will be convenient to define

$$\delta = \begin{cases} 1 & \text{if } p = 2 \\ 0 & \text{if } p \neq 2. \end{cases}$$

Proposition 1.10 *Suppose that ϕ is supported on $p^m\mathbb{Z}_p$ and constant on $p^n\mathbb{Z}_p$-cosets for $m \leq n$. Then, $\widehat{\phi}$ is supported on $p^{-n+(c-\delta)}\mathbb{Z}_p$ and constant on $p^{-m+(c-\delta)}\mathbb{Z}_p$-cosets.*

Proof: Since ϕ is constant on $p^n\mathbb{Z}_p$-cosets,

$$\begin{aligned}\widehat{\phi}(y) &= \int_{\mathbb{Q}_p} \psi(2uy)\phi(u)du \\ &= \int_{\mathbb{Q}_p} \psi(2uy)\phi(u+p^n)du \\ &= \int_{\mathbb{Q}_p} \psi(2(v-p^n)y)\phi(v)dv \\ &= \psi(-2p^n y)\widehat{\phi}(y)\end{aligned}$$

so $\widehat{\phi}(y) \neq 0$ if and only if $y \in p^{-n+(c-\delta)}\mathbb{Z}_p$.

The Fourier transform $\widehat{\phi}$ is constant on $p^r\mathbb{Z}_p$-cosets if and only if r is the smallest integer such that $\psi(2p^r u) = 1$ for all $u \in p^m\mathbb{Z}_p$, which, in turn, happens if and only if $r = -m + (c-\delta)$. ∎

Proposition 1.11 *For $m \in \mathbb{Z}$, denote the characteristic function of $p^m\mathbb{Z}_p$ by ϕ_m. Then*

$$\widehat{\phi}_m = p^{-m+(c-\delta)/2}\phi_{-m+(c-\delta)}$$

Proof: The Fourier transform of $\widehat{\phi}_m$ is given by

$$\widehat{\phi}_m(y) = \int_{p^m\mathbb{Z}_p} \psi(2uy)du = \begin{cases} \mathrm{vol}(p^m\mathbb{Z}_p) & \text{if } y \in p^{-m+(c-\delta)}\mathbb{Z}_p \\ 0 & \text{otherwise.} \end{cases}$$

The normalization of the Haar measure on $\mathbb{Q}_p$ gives

$$\phi_0 = \widehat{\widehat{\phi}}_0 = \mathrm{vol}(\mathbb{Z}_p)\widehat{\phi}_{c-\delta} = \mathrm{vol}(\mathbb{Z}_p)\,\mathrm{vol}(p^{c-\delta}\mathbb{Z}_p)\phi_0 = p^{-(c-\delta)}\,\mathrm{vol}(\mathbb{Z}_p)^2\phi_0,$$

so the measure (with respect to ψ) of $\mathbb{Z}_p$ is $p^{-(c-\delta)/2}$. ∎

Corollary 1.12 *If ψ is a character of conductor $c = \delta$, then the Fourier transform with respect to ψ is given by $\widehat{\phi}_m = p^{-m}\phi_{-m}$.*

1.7.1 Functions on $\mathbb{Q}_p^n$

Suppose that $\mathcal{Y}$ is a vector space over $\mathbb{Q}_p$ and consider elements of $\mathcal{Y}$ as column vectors with respect to some basis. Let $S(\mathcal{Y})$ denote the set of Schwarz functions on $\mathcal{Y}$ and fix an additive character ψ of $\mathbb{Q}_p$. The Fourier transform with respect to ψ on $S(\mathcal{Y})$ is defined by $\phi \mapsto \widehat{\phi}$ where

$$\widehat{\phi}(y) = \int_{\mathcal{Y}} \psi(2\,{}^{\intercal}uy)\phi(u)du.$$

The Haar measure on $\mathcal{Y}$ is also normalized to give $\widehat{\widehat{\phi}}(-y) = \phi(y)$.

1.8 Weil Representation

Let F be a nonarchimedian local field with absolute value $|\cdot|$, and let $\mathcal{W}$ be a $2n$-dimensional symplectic space over F with symplectic form Q. Fix an additive character ψ of F.

The Heisenberg group $H(\mathcal{W})$ is defined to be the set $\mathcal{W} \times F$ with group multiplication

$$(u,s)\cdot(v,t) = \big(\, u+v\,,\ s+t+Q(u,v)\big).$$

The center of $H(\mathcal{W})$ is $Z = \{(0,t)\} = F$; indeed, $(v,t) \in Z$ if and only if $Q(u,v) = 0$ for every u in $\mathcal{W}$. Since the symplectic group $\mathrm{Sp}(\mathcal{W})$ preserves Q, it acts as a group of automorphisms of the Heisenberg group by

$$g(v,t) = (gv,t).$$

Let (ρ, S) be a representation of $H(\mathcal{W})$ with central character ψ; that is, $\rho(0,t)h = \psi(t)h$. One can twist ρ by $g \in \mathrm{Sp}(\mathcal{W})$ to obtain the representation (ρ^g, S) given by

$$\rho^g(v,t) = \rho\big(g(v,t)\big) = \rho(gv,t),$$

which also has central character ψ. By the Stone-von Neumann theorem, ρ^g must be isomorphic to ρ, so for each $g \in \mathrm{Sp}(\mathcal{W})$, there exists an operator $T(g) : S \to S$, unique up to a scalar in $\mathbb{C}^\times$, which intertwines the two representations; that is,

$$T(g)\rho = \rho^g T(g).$$

This operator $T : \mathrm{Sp}(\mathcal{W}) \to \mathrm{GL}(S)/\mathbb{C}^\times$ is a projective representation of $\mathrm{Sp}(\mathcal{W})$. Define a subgroup $\widetilde{\mathrm{Sp}}(\mathcal{W})$ of $\mathrm{Sp}(\mathcal{W}) \times \mathrm{GL}(S)$ by

$$\widetilde{\mathrm{Sp}}(\mathcal{W}) = \Big\{\big(g, T(g)\big) : T(g)\rho = \rho^g T(g)\Big\},$$

which fits into the exact sequence

$$1 \to \mathbb{C}^\times \to \widetilde{\mathrm{Sp}}(\mathcal{W}) \to \mathrm{Sp}(\mathcal{W}) \to 1.$$

The group $\widetilde{\mathrm{Sp}}(\mathcal{W})$ is a central extension of $\mathrm{Sp}(\mathcal{W})$ by $\mathbb{C}^\times$, so T lifts (see [9]) to a linear representation $\omega : \widetilde{\mathrm{Sp}}(\mathcal{W}) \to \mathrm{GL}(S)$, called the *Weil representation* with respect to ψ.

1.8.1 Models

For any closed subgroup $\mathcal{Z}$ of $\mathcal{W}$ define

$$\mathcal{Z}^{\perp} = \{v \in \mathcal{W} : \psi\big(Q(v,z)\big) = 1 \text{ for all } z \in \mathcal{Z}\},$$

which is also a closed subgroup of $\mathcal{W}$. In addition, define $H(\mathcal{Z})$ to be the subgroup $\mathcal{Z} \times F$ of the Heisenberg group. Assume now that $\mathcal{Z} \subset \mathcal{Z}^{\perp}$. The character ψ can be extended trivially to $H(\mathcal{Z})$ by $\psi(z,t) = \psi(t)$; indeed,

$$\begin{aligned} \psi\big((z_1,t_1)\cdot(z_2,t_2)\big) &= \psi\big(z_1+z_2, t_1+t_2+Q(z_1,z_2)\big) \\ &= \psi(t_1)\psi(t_2)\psi\big(Q(z_1,z_2)\big) \\ &= \psi(z_1,t_1)\psi(z_2,t_2). \end{aligned}$$

Define $(\rho, S_{\mathcal{Z}})$ to be the induced representation $\mathrm{ind}_{H(\mathcal{Z})}^{H(\mathcal{W})}\psi$ of $H(\mathcal{W})$; that is,

$$S_{\mathcal{Z}} = \left\{ f \in C^{\infty}\big(H(\mathcal{W})\big) \ : \ \begin{array}{l} \text{1. } f \text{ has compact support modulo } H(\mathcal{Z}), \\ \text{2. } f(zh) = \psi(z)f(h) \text{ for all } z \in H(\mathcal{Z}), h \in H(\mathcal{W}), \\ \text{3. there exists an open compact subgroup } K_f \text{ of} \\ \quad H(\mathcal{W}) \text{ such that } f(hk) = f(h) \text{ for all } k \in K_f \end{array} \right\}$$

and ρ acts on $S_{\mathcal{Z}}$ by right translation $\big(\rho(h_1)f\big)(h_2) = f(h_2h_1)$. The restriction of ρ to the center of $H(\mathcal{W})$ is given by

$$\big(\rho(0,t)f\big)(h) = f\big(h\cdot(0,t)\big) = f\big((0,t)\cdot h\big) = \psi(t)f(h);$$

hence, ρ has central character ψ.

For closed subgroups $\mathcal{Z}$ of $\mathcal{W}$, one can build a model for ρ and hence a model for ω. If $\mathcal{Z}$ is a maximal isotropic subspace of $\mathcal{W}$, the model is called Schrödinger's model.

1.8.2 Schrödinger's Model

Let $\mathcal{X} + \mathcal{Y}$ be a complete polarization of $\mathcal{W}$. As $\mathcal{X}$ is a maximal isotropic subspace of $\mathcal{W}$, Q is identically zero on $\mathcal{X}$, and hence $\mathcal{X} = \mathcal{X}^{\perp}$. Following the construction above, one obtains the space of functions $S_{\mathcal{X}}$. The space $S_{\mathcal{X}}$ is canonically isomorphic to the space $S(\mathcal{Y})$ of Schwarz functions on $\mathcal{Y}$ via the map $S_{\mathcal{X}} \to S(\mathcal{Y})$ given by $f \mapsto \phi$, where

$$\phi(y) = f(y,0).$$

Proposition 1.13 *In Schrödinger's model, the representation* ρ *takes the form*

$$\begin{aligned}\big(\rho(x,0)\phi\big)(y_0) &= \psi\big(-2Q(x,y_0)\big)\phi(y_0)\\ \big(\rho(y,0)\phi\big)(y_0) &= \phi(y+y_0)\\ \big(\rho(0,t)\phi\big)(y_0) &= \psi(t)\phi(y_0)\end{aligned}$$

Proof: Let $t \in F$, $x \in \mathcal{X}$, and $y_0, y \in \mathcal{Y}$. Then

$$\begin{aligned}\big(\rho(x+y,t)\phi\big)(y_0) &= f\big((y_0,0)\cdot(x+y,t)\big)\\ &= f\big(x+y+y_0\,,\, t+Q(y_0,x+y)\big)\\ &= f\big((x,t+Q(y_0,x)-Q(x,y+y_0))\cdot(y+y_0,0)\big)\\ &= \psi\big(x,t-Q(x,y+2y_0)\big)f(y+y_0,0)\\ &= \psi\big(t-Q(x,y+2y_0)\big)\phi(y+y_0).\end{aligned}$$

■

The next step is to find the intertwining operator T. The group $\mathrm{Sp}(\mathcal{W})$, under the symplectic basis corresponding to the polarization $\mathcal{X}+\mathcal{Y}$, is generated by the following types of matrices:

$$\begin{aligned}\underline{x}(b) &= \begin{bmatrix} 1 & b \\ & 1 \end{bmatrix} \quad (b \text{ a symmetric } n\times n \text{ matrix}),\\ \underline{h}(b) &= \begin{bmatrix} b & \\ & {}^{\intercal}b^{-1} \end{bmatrix} \quad (b \text{ an invertible } n\times n \text{ matrix}),\\ \underline{w} &= \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}.\end{aligned}$$

Proposition 1.14 *The intertwining map* $T : \mathrm{Sp}(\mathcal{W}) \to \mathrm{GL}\big(S(\mathcal{Y})\big)/\mathbb{C}^\times$ *is given by*

$$\begin{aligned}T\big(\underline{x}(b)\big)\phi(y) &= \psi({}^{\intercal}yby)\phi(y)\\ T\big(\underline{h}(b)\big)\phi(y) &= \phi({}^{\intercal}by)\\ T(\underline{w})\phi(y) &= \widehat{\phi}(y),\end{aligned}$$

where $\widehat{\phi}$ *denotes the Fourier transform of* ϕ *with respect to the fixed character* ψ.

Proof: Note that the action of $(0,t)$ under ρ will clearly be intertwined by these operators, and hence, it remains only to verify the proposition for $\rho_0(x,y) = \rho(x+y,0)$. First, the formula $T(g)\rho = \rho^g T(g)$ is verified for $g = \underline{x}(b)$:

$$\begin{aligned}
\rho_0^{\underline{x}(b)}(x,y)\Big(T\big(\underline{x}(b)\big)\phi\Big)(y_0) &= \rho_0(x,bx+y)\Big(T\big(\underline{x}(b)\big)\phi\Big)(y_0)\\
&= \psi\big(-{}^{\mathsf{T}}(x+by)(y+2y_0)\big)\Big(T\big(\underline{x}(b)\big)\phi\Big)(y+y_0)\\
&= \psi\big(-{}^{\mathsf{T}}(x+by)(y+2y_0)\big)\psi\big({}^{\mathsf{T}}(y+y_0)b(y+y_0)\big)\phi(y+y_0)\\
&= \psi({}^{\mathsf{T}}y_0by_0)\psi\big(-{}^{\mathsf{T}}x(y+2y_0)\big)\phi(y+y_0)\\
&= \psi({}^{\mathsf{T}}y_0by_0)\big(\rho_0(x,y)\phi\big)(y_0)\\
&= T\big(\underline{x}(b)\big)\big(\rho_0(x,y)\phi\big)(y_0).
\end{aligned}$$

Next, it is verified for $g = \underline{h}(b)$:

$$\begin{aligned}
\rho_0^{\underline{h}(b)}(x,y)\Big(T\big(\underline{h}(b)\big)\phi\Big)(y_0) &= \rho_0(bx,{}^{\mathsf{T}}b^{-1}y)\Big(T\big(\underline{h}(b)\big)\phi\Big)(y_0)\\
&= \psi\big(-{}^{\mathsf{T}}x(y+2\,{}^{\mathsf{T}}by_0)\big)\Big(T\big(\underline{h}(b)\big)\phi\Big)(y_0)\\
&= \psi\big(-{}^{\mathsf{T}}x(y+2\,{}^{\mathsf{T}}by_0)\big)\phi(y+{}^{\mathsf{T}}by_0)\\
&= \big(\rho_0(x,y)\phi\big)({}^{\mathsf{T}}by_0)\\
&= T\big(\underline{h}(b)\big)\big(\rho_0(x,y)\phi\big)(y_0).
\end{aligned}$$

Finally, it is verified for $g = \underline{w}$:

$$\begin{aligned}
\rho_0^{\underline{w}}(x,y)\big(T(\underline{w})\phi\big)(y_0) &= \rho_0(y,-x)\widehat{\phi}(y_0)\\
&= \psi\big(-{}^{\mathsf{T}}y(-x+2y_0)\big)\widehat{\phi}(-x+y_0)\\
&= \psi\big(-{}^{\mathsf{T}}y(-x+2y_0)\big)\int_{\mathcal{Y}}\psi\big(2\,{}^{\mathsf{T}}u(-x+y_0)\big)\phi(u)du\\
&= \psi\big(-{}^{\mathsf{T}}y(-x+2y_0)\big)\int_{\mathcal{Y}}\psi\big(2\,{}^{\mathsf{T}}(v+y)(-x+y_0)\big)\phi(v+y)dv\\
&= \int_{\mathcal{Y}}\psi(2\,{}^{\mathsf{T}}vy_0)\psi\big(-{}^{\mathsf{T}}x(y+2v)\big)\phi(v+y)dv\\
&= \int_{\mathcal{Y}}\psi(2\,{}^{\mathsf{T}}vy_0)\big(\rho_0(x,y)\phi\big)(v)dv\\
&= T(\underline{w})\big(\rho_0(x,y)\phi\big)(y_0).
\end{aligned}$$

∎

Considering $\widetilde{\mathrm{Sp}}(\mathcal{W})$ as a central extension of the symplectic Chevalley group, take $x(b)$, $h(b)$ and w to be the canonical lifts of $\underline{x}(b)$, $\underline{h}(b)$, and $\underline{w}$, as in Section 1.4. The Schrödinger model of the Weil representation ω is given by

$$\begin{aligned}
x(b)\phi(y) &= \alpha_b\psi({}^{\mathsf{T}}yby)\phi(y)\\
h(b)\phi(y) &= \beta_b|\det(b)|^{1/2}\phi({}^{\mathsf{T}}by)\\
w\phi(y) &= \gamma_1\widehat{\phi}(y)
\end{aligned}$$

for some constants $\alpha_b, \beta_b, \gamma_1 \in \mathbb{C}^\times$. The factor $|\det(b)|^{1/2}$ is a normalization term that is included for convenience.

The Weil representation is not irreducible. Since $\psi\big({}^\intercal(-y)b(-y)\big) = \psi({}^\intercal yby)$, and since the Fourier transform of an even function is even, the space of even functions is an ω-invariant subspace of $S(\mathcal{Y})$, called the even Weil representation. Similarly, the space of odd functions is called the odd Weil representation.

CHAPTER 2

WEIL REPRESENTATION OF $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$

Throughout this chapter, $\underline{G} = \mathrm{SL}_2(\mathbb{Q}_p)$, $G = \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ is a central extension of $\underline{G}$ by $\mathbb{C}^\times$, $\psi = \psi_a$ is an additive character of $\mathbb{Q}_p$ of conductor $c = -\operatorname{val}(a)$, and $V = S(\mathbb{Q}_p)$ is the Schrödinger model of the Weil representation of G associated to ψ. In addition, ϕ_m will denote the characteristic function of $p^m\mathbb{Z}_p$.

The group $\underline{G}$, as a Chevalley group, is generated by the elements

$$\underline{x}(t) = \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}, \qquad \text{where } t \in \mathbb{Q}_p,$$
$$\underline{y}(t) = \begin{bmatrix} 1 & \\ t & 1 \end{bmatrix}, \qquad \text{where } t \in \mathbb{Q}_p.$$

For $t \in \mathbb{Q}_p^\times$, the elements $\underline{w}(t)$ and $\underline{h}(t)$ are defined by

$$\underline{w}(t) = \underline{x}(t)\underline{y}(-1/t)\underline{x}(t) = \begin{bmatrix} & t \\ -t^{-1} & \end{bmatrix},$$

$$\underline{h}(t) = \underline{w}(t)\underline{w}(-1) = \begin{bmatrix} t & \\ & t^{-1} \end{bmatrix}.$$

Let $x(t)$ and $y(t)$ be the unique lifts of $\underline{x}(t)$ and $\underline{y}(t)$; these elements generate the central extension G of $\underline{G}$. For $t \in \mathbb{Q}_p^\times$, define the elements $w(t)$ and $h(t)$ of G by

$$w(t) = x(t)y(-1/t)x(t), \qquad \text{and} \qquad h(t) = w(t)w(-1).$$

These are the canonical lifts of $\underline{w}(t)$ and $\underline{h}(t)$, respectively.

The elements $w_0 = w(1/p)$ and $w_1 = w(1)$ form a set of representatives in G of the generators $\{s_0, s_1\}$ of the affine Weyl group W^{aff}. The full set of representatives of W^{aff} in G is $\{w(p^n), h(p^n) : n \in \mathbb{Z}\}$.

The Schrödinger model ω of the Weil representation of G on V is given by

$$x(t)\phi(y) = \alpha_t \psi(ty^2)\phi(y)$$
$$h(t)\phi(y) = \beta_t |t|^{1/2}\phi(ty)$$
$$w_1\phi(y) = \gamma_1\widehat{\phi}(y),$$

where the Fourier transform $\widehat{\phi}$ of ϕ is as in Section 1.7. It will be useful to consider the action of $w(t)$ which, for $\gamma_t = \beta_t\gamma_1$, is given by

$$w(t)\phi(y) = \gamma_t|t|^{1/2}\widehat{\phi}(ty).$$

2.1 The Constants α_t, β_t, and γ_t

Proposition 2.1 *The constant α_t is equal to 1 for all $t \in \mathbb{Q}_p$.*

Proof: In any central extension of $\mathrm{SL}_2(\mathbb{Q}_p)$, one has $h(r)x(s)h(r)^{-1} = x(sr^2)$, and hence,

$$\big[h(r), x(s)\big] = x\big(s(r^2-1)\big).$$

Since $\mathbb{Q}_p^\times$ has an element whose square is not equal to 1, then every $x(t)$ can be expressed as a commutator $x(t) = [h(r), x(s)]$ where $t = s(r^2-1)$. Therefore, for any $\phi \in V$,

$$\alpha_t\psi(ty^2)\phi(y) = x(t)\phi(y) = h(r)x(s)h(r^{-1})x(-s)\phi(y) = \psi\big(s(r^2-1)y^2\big)\phi(y),$$

so $\alpha_t = 1$. This fact is independent of the choice of ψ. ∎

Proposition 2.2 *Let t be an element of $\mathbb{Q}_p^\times$ with $n = \mathrm{val}(t)$. Let m be the integer such that $n-c-1 \leq 2m \leq n-c$, and set $\ell = m+(c-\delta)$ with δ as in Proposition 1.11 so that $\widehat{\phi}_{-m} = \mathrm{vol}(p^{-m}\mathbb{Z}_p)\phi_\ell$. Then,*

$$\gamma_t = p^{n/2}\int_{p^\ell\mathbb{Z}_p}\psi(u^2/t)du.$$

Proof: On the one hand,

$$w(t)\phi_{-m}(0) = \gamma_t p^{-n/2}\,\mathrm{vol}(p^{-m}\mathbb{Z}_p)\phi_\ell(0) = \gamma_t p^{-n/2}\,\mathrm{vol}(p^{-m}\mathbb{Z}_p).$$

On the other hand, in G, one has

$$w(t) = x(t)y(-1/t)x(t) = x(t)w_1x(1/t)w_1^{-1}x(t),$$

so $w(t)\phi_{-m}(0)$ may be computed using

$$\begin{aligned}
w(t)\phi_{-m}(y) &= x(t)w_1x(1/t)w_1^{-1}x(t)\phi_{-m}(y)\\
&= x(t)w_1x(1/t)w_1^{-1}\phi_{-m}(y)\\
&= \gamma_1^{-1}\,\mathrm{vol}(p^{-m}\mathbb{Z}_p)x(t)w_1x(1/t)\phi_\ell(y)\\
&= \gamma_1^{-1}\,\mathrm{vol}(p^{-m}\mathbb{Z}_p)x(t)w_1\big(\psi(y^2/t)\phi_\ell(y)\big)\\
&= \mathrm{vol}(p^{-m}\mathbb{Z}_p)\psi(ty^2)\int_{p^\ell\mathbb{Z}_p}\psi(u^2/t)\psi(2uy)du;
\end{aligned}$$

that is,

$$w(t)\phi_{-m}(0) = \mathrm{vol}(p^{-m}\mathbb{Z}_p)\int_{p^\ell\mathbb{Z}_p}\psi(u^2/t)du.$$

■

Corollary 2.3 *For any $s \in \mathbb{Q}_p^\times$, let s' in $\mathbb{Z}_p^\times$ be given by $s' = p^{-\mathrm{val}(s)}s$. The constant γ_t is given as follows.*

1. *For $p = 2$,*

$$\gamma_t = \begin{cases} \psi_1(a't'/8) & \textit{if } \mathrm{val}(a), \mathrm{val}(t) \textit{ different parity,} \\ \begin{cases} \psi_1(1/8) & \textit{if } a' \equiv t' \ (4) \\ \psi_1(7/8) & \textit{otherwise} \end{cases} & \textit{if } \mathrm{val}(a), \mathrm{val}(t) \textit{ same parity.} \end{cases}$$

2. *For $p \neq 2$,*

$$\gamma_t = \begin{cases} 1 & \textit{if } \mathrm{val}(a), \mathrm{val}(t) \textit{ different parity,} \\ \begin{cases} \left(\frac{a't'}{p}\right) & \textit{if } p \equiv 1 \ (4) \\ \left(\frac{a't'}{p}\right) i & \textit{if } p \equiv 3 \ (4) \end{cases} & \textit{if } \mathrm{val}(a), \mathrm{val}(t) \textit{ same parity.} \end{cases}$$

Proof: This can be computed explicitly using the previous proposition in the four cases, recalling that $\delta = 1$ for $p = 2$ and $\delta = 0$ for $p \neq 2$.

Case 1 ($p = 2$, $n = 2m + c + 1$): If $u \in 2^\ell x + 2^{\ell+3}\mathbb{Z}_2$, then $u^2 \in 2^{n+c}(x^2/8 + \mathbb{Z}_2)$, and hence, $\psi_a(u^2/t) = \psi_{a'/t'}(x^2/8)$, which depends on the values of a' and t' modulo 8. Therefore,

$$\begin{aligned} \gamma_t &= 2^{n/2}\sum_{x\in\mathbb{Z}/8\mathbb{Z}}\int_{2^\ell x + 2^{\ell+3}\mathbb{Z}_2}\psi(u^2/t)du \\ &= 2^{n/2}\,\mathrm{vol}(2^{\ell+3}\mathbb{Z}_2)\sum_{x\in\mathbb{Z}/8\mathbb{Z}}\psi_{a'/t'}(x^2/8) \\ &= 2^{-2}\big(4\psi_{a'/t'}(1/8)\big) \\ &= \psi_{a'/t'}(1/8). \end{aligned}$$

Every invertible element in $\mathbb{Z}/8\mathbb{Z}$ has order 2, so $\psi_{a'/t'}(1/8) = \psi_{a't'}(1/8)$.

Case 2 ($p = 2$, $n = 2m + c$): If $u \in 2^\ell x + 2^{\ell+2}\mathbb{Z}_2$, then $u^2 \in 2^{n+c}(x^2/4 + \mathbb{Z}_2)$, and hence, $\psi_a(u^2/t) = \psi_{a'/t'}(x^2/4)$, which depends on the values of a' and t' modulo 4. Therefore,

$$\begin{aligned}
\gamma_t &= 2^{n/2} \sum_{x\in\mathbb{Z}/4\mathbb{Z}} \int_{2^\ell x + 2^{\ell+2}\mathbb{Z}_2} \psi(u^2/t)du \\
&= 2^{n/2}\operatorname{vol}(2^{\ell+2}\mathbb{Z}_2) \sum_{x\in\mathbb{Z}/4\mathbb{Z}} \psi_{a'/t'}(x^2/4) \\
&= 2^{-3/2}\big(2 + 2\psi_{a'/t'}(1/4)\big) \\
&= \frac{1}{\sqrt{2}} + \frac{\psi_{a'/t'}(1/4)}{\sqrt{2}} \\
&= \begin{cases} \psi_1(1/8) & \text{if } a' \equiv t' \ (4) \\ \psi_1(7/8) & \text{otherwise.} \end{cases}
\end{aligned}$$

Case 3 ($p \neq 2$, $n = 2m + c + 1$): If $u \in p^\ell x + p^{\ell+1}\mathbb{Z}_p$, then $u^2 \in p^{n+c}(x^2/p + \mathbb{Z}_p)$, and hence, $\psi_a(u^2/t) = \psi_{a'/t'}(x^2/p)$, which depends on the values of a' and t' modulo p. Therefore,

$$\begin{aligned}
\gamma_t &= p^{n/2} \sum_{x\in\mathbb{F}_p} \int_{p^\ell x + p^{\ell+1}\mathbb{Z}_p} \psi(u^2/t)du \\
&= p^{n/2}\operatorname{vol}(p^{\ell+1}\mathbb{Z}_p) \sum_{x\in\mathbb{F}_p} \psi_{a'/t'}(x^2/p) \\
&= p^{-1/2}\Big[\psi_{a'/t'}(0) + 2\sum_{x\in S} \psi_{a'/t'}(x/p) - \sum_{x\in\mathbb{F}_p} \psi_{a'/t'}(x/p)\Big] \\
&= p^{-1/2} \sum_{x\in\mathbb{F}_p^\times} \left(\frac{x}{p}\right) \psi_{a'/t'}(x/p).
\end{aligned}$$

The set S appearing in the calculation above is the set of squares in $\mathbb{F}_p$. The sum in the final line is a quadratic Gauss sum, and, since

$$\sum_{x\in\mathbb{F}_p^\times} \left(\frac{x}{p}\right) \psi_1(x/p) = \begin{cases} \sqrt{p} & \text{if } p \equiv 1 \ (4) \\ i\sqrt{p} & \text{if } p \equiv 3 \ (4), \end{cases}$$

a change of variables implies that

$$\gamma_t = \begin{cases} \left(\frac{a'/t'}{p}\right) & \text{if } p \equiv 1 \ (4) \\ \left(\frac{a'/t'}{p}\right) i & \text{if } p \equiv 3 \ (4). \end{cases}$$

The set S of squares in $\mathbb{F}_p$ is closed under inversion, so $\left(\dfrac{a'/t'}{p}\right) = \left(\dfrac{a't'}{p}\right)$.

Case 4 ($p \neq 2$, $n = 2m + c$): If $u \in p^{\ell}\mathbb{Z}_p$, then $u^2 \in p^{n+c}\mathbb{Z}_p$, and hence, $\psi_a(u^2/t) = 1$. Therefore,

$$\gamma_t = p^{n/2}\,\mathrm{vol}(p^{\ell}\mathbb{Z}_p) = 1.$$

■

Corollary 2.4 *If $a = p^{-c}$, then γ_1 (the Weil index associated to $\psi = \psi_a$) is given by*

$$\gamma_1 = \begin{cases} \psi_1(1/8) & \text{if } p = 2, \\ i & \text{if } p \equiv 3\ (4) \text{ and } c \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases}$$

Corollary 2.5 *The constant β_t is a 4th root of unity, so G could be taken to be a two-fold central extension of $\underline{G}$. Moreover, the symbol (t,u), given by $\beta_t\beta_u = (t,u)\beta_{tu}$, is the Hilbert symbol.*

Proof: Since $\beta_t = \gamma_t/\gamma_1$, this corollary follows from the fact that γ_t is a primitive 8th root of unity for $p = 2$ and a 4th root of unity for $p \neq 2$. See [6] for details on the symbol (t,u).
■

2.2 Iwahori-fixed Vectors

Let $\underline{I}$ be the Iwahori subgroup of $\mathrm{SL}_2(\mathbb{Z}_p)$ given by

$$\underline{I} = \left\{ \begin{bmatrix} a & b \\ pc & d \end{bmatrix} : a,b,c,d \in \mathbb{Z}_p \right\},$$

and let I be its full inverse image in $G = \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$. Then, I is generated by the elements

$$\{h(t) : t \in \mathbb{Z}_p^{\times}\} \cup \{x(t) : t \in \mathbb{Z}_p\} \cup \{y(t) : t \in p\mathbb{Z}_p\}.$$

Suppose that ϕ is an element of the Weil representation V that is supported on $p^m\mathbb{Z}_p$ and constant on $p^n\mathbb{Z}_p$-cosets. For ϕ to be a fixed vector under the action of I, it must be true that $x(t)\phi = \phi$ for all $t \in \mathbb{Z}_p$ and $x(t)\widehat{\phi} = \widehat{\phi}$ for all $t \in p\mathbb{Z}_p$. According to Proposition 1.10, $\widehat{\phi}$ is supported on $p^{-n+(c-\delta)}\mathbb{Z}_p$. Since $x(t)$ acts by $\psi(ty^2)$, ϕ can be fixed under I only if

$$\left\{ \begin{array}{l} \psi(ty^2) = 1 \text{ for } t \in \mathbb{Z}_p, y \in p^m\mathbb{Z}_p \\ \psi(ty^2) = 1 \text{ for } t \in p\mathbb{Z}_p, y \in p^{-n+(c-\delta)}\mathbb{Z}_p \end{array} \right\}, \text{ i.e., if } c \leq 2m \leq 2n \leq c+1-2\delta.$$

In particular, there are no Iwahori-fixed vectors for $p = 2$.

2.3 A Minimal Type for $p = 2$

For the remainder of the section, the discussion will be restricted to the case $p = 2$ with the additive character $\psi = \psi_{1/2}$ of $\mathbb{Q}_2$ which is trivial on $2\mathbb{Z}_2$. In this setting, the Fourier transform on ϕ_m is given by $\widehat{\phi}_m = 2^{-m}\phi_{-m}$, and the volume of $\mathbb{Z}_2$ is equal to 1.

Let K be the open compact subgroup of $I \subset G$ generated by

$$\{h(t) : t \in \mathbb{Z}_2^\times\} \cup \{x(t) : t \in 2\mathbb{Z}_2\} \cup \{y(t) : t \in 2\mathbb{Z}_2\};$$

that is, K is the full inverse image of the group

$$\underline{K} = \left\{ \begin{bmatrix} a & 2b \\ 2c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Q}_2) : a, b, c, d \in \mathbb{Z}_2 \right\}.$$

The subspace $\mathbb{C}\phi_0$ of V is K-stable under the Weil representation since

$$\begin{aligned} x(2)\phi_0 &= \phi_0, \\ h(t)\phi_0 &= \beta_t\phi_0 \text{ for } t \in \mathbb{Z}_2^\times, \\ w_1\phi_0 &= \gamma_1\phi_0. \end{aligned}$$

Denote by $\bar{\chi} : K \to \mathbb{C}^\times$ the character satisfying $\omega(k)\phi_0 = \bar{\chi}(k)\phi_0$ for $k \in K$ which acts on the nontrivial subspace

$$V^{K,\bar{\chi}} = \{\phi \in V : \omega(k)\phi = \bar{\chi}(k)\phi \text{ for all } k \in K\}.$$

Proposition 2.6 *The character $\bar{\chi}$ satisfies the following.*

1. $\bar{\chi}(x(t)) = \bar{\chi}(y(t)) = 1$ *for* $t \in 2\mathbb{Z}_2$.
2. $\bar{\chi}(x(-1)y(t)x(1)) = \bar{\chi}(x(1)y(t)x(-1)) = \psi(-t/4)$ *for* $t \in 2\mathbb{Z}_2$.
3. $V^{K,\bar{\chi}} = \mathbb{C}\phi_0$.

Proof: Since $\widehat{\phi}_0 = \phi_0$ and since $y(t) = w_1 x(-t) w_1^{-1}$, in order to prove the first part, it suffices to compute the action of $\bar{\chi}(x(t))$ on ϕ_0 given by

$$\bar{\chi}(x(t))\phi_0(y) = \psi(ty^2)\phi_0(y).$$

Clearly, this acts as 1 on ϕ_0 if and only if $t \in 2\mathbb{Z}_2$.

By the first part of the proposition, $\bar{\chi}\big(x(\pm 2)\big) = 1$, so

$$\bar{\chi}\big(x(-1)y(t)x(1)\big) = \bar{\chi}\big(x(2)\big)\bar{\chi}\big(x(-1)y(t)x(1)\big)\bar{\chi}\big(x(-2)\big) = \bar{\chi}\big(x(1)y(t)x(-1)\big).$$

To prove the second part of the proposition, it then suffices to compute the action of

$$x(1)y(t)x(-1) = x(1)w_1x(-t)w_1^{-1}x(-1)$$

on ϕ_0. The action of $x(-1)$ on ϕ_0 is given by

$$x(-1)\phi_0(y) = \psi(-y^2)\phi_0(y) = \left\{ \begin{array}{rl} 1 & \text{if } y \in 2\mathbb{Z}_2 \\ -1 & \text{if } y \in \mathbb{Z}_2 \setminus 2\mathbb{Z}_2 \\ 0 & \text{otherwise} \end{array} \right\} = \big(-\phi_0 + 2\phi_1\big)(y).$$

Therefore,

$$\begin{aligned} x(1)w_1x(-t)w_1^{-1}\big(x(-1)\phi_0\big)(y) &= x(1)w_1x(-t)w_1^{-1}\big(-\phi_0 + 2\phi_1\big)(y) \\ &= \gamma_1^{-1}x(1)w_1\big(\psi(-ty^2)(-\phi_0 + \phi_{-1})\big)(y) \\ &= x(1)\int_{\mathbb{Q}_2}\psi(2uy)\psi(-tu^2)(-\phi_0 + \phi_{-1})(u)du \\ &= \psi(y^2)\int_{1/2+\mathbb{Z}_2}\psi(2uy)\psi(-tu^2)du, \end{aligned}$$

the last equality following from the fact that $-\phi_0 + \phi_{-1}$ is equal to the characteristic function of $1/2 + \mathbb{Z}_2$. Evaluating this expression at $y = 0$ gives

$$x(1)y(t)x(-1)\phi_0(0) = \int_{1/2+\mathbb{Z}_2}\psi(-tu^2)du.$$

If $u \in 1/2 + \mathbb{Z}_2$, then $u^2 \in 1/4 + \mathbb{Z}_2$, so for $t \in 2\mathbb{Z}_2$, $\psi(-tu^2) = \psi(-t/4)$. Since $\mathrm{vol}(\mathbb{Z}_2) = 1$, one has

$$x(1)y(t)x(-1)\phi_0(0) = \psi(-t/4).$$

To prove the third part, it is enough to show that $V^{K,\bar{\chi}} \subset \mathbb{C}\phi_0$. Let ϕ be an arbitrary element of $V^{K,\bar{\chi}}$ which is supported on $2^m\mathbb{Z}_2$ and constant on $2^n\mathbb{Z}_2$-cosets for $m \leq n$. Then, $\widehat{\phi}$ is supported on $2^{-n}\mathbb{Z}_2$. Since $x(2)$ and $y(2)$ act trivially on $V^{K,\bar{\chi}}$, one must have that $\psi(2y^2) = 1$ for all $y \in 2^m\mathbb{Z}_2$ and all $y \in 2^{-n}\mathbb{Z}_2$. Therefore, $0 \leq m \leq n \leq 0$, so $\phi \in \mathbb{C}\phi_0$. ∎

2.4 The Associated Hecke Algebra

The Hecke algebra $\mathcal{H} = \mathcal{H}(G/\!/K;\chi)$ for this even type of the Weil representation is

$$\mathcal{H} = \left\{ f \in C_c^\infty(G) : f(k_1 x k_2) = \chi(k_1) f(x) \chi(k_2) \text{ for all } k_i \in K, x \in G \right\}$$

with convolution

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1} g) dx.$$

An element f of $\mathcal{H}$ is determined by its value on a set of representatives of $K\backslash G/K$ or, equivalently, of $\underline{K}\backslash\underline{G}/\underline{K}$. By [5],

$$\underline{G} = \mathrm{SL}_2(\mathbb{Q}_2) = \bigsqcup_{s \in W^{\mathrm{aff}}} \underline{I}\ \underline{w}\ \underline{I}$$

where $\underline{I}$ is the Iwahori subgroup, W^{aff} is the affine Weyl group, and $\underline{w}$ is a representative of s in $\underline{G}$.

It is straight-forward to see that $\underline{K}$ is a normal subgroup of $\underline{I}$ and that $\underline{I}/\underline{K}$ and $\underline{K}\backslash\underline{I}$ are each isomorphic to the group $B(2) = \{\underline{x}(t) : t \in \mathbb{F}_2\}$. A set of double coset representatives of $\underline{K}\backslash\underline{G}/\underline{K}$, and hence of $K\backslash G/K$, must then be contained in the set $B(2)\ W^{\mathrm{aff}}\ B(2)$. However, some of these representatives are redundant and will now be eliminated. Working in the central extension G, since $h(t)x(1)h(t)^{-1} = x(t^2)$,

$$\begin{aligned} Kx(1)h(2^n)K &= Kh(2^n)x(2^{-2n})K = Kh(2^n)K \text{ for } n < 0, \\ Kh(2^n)x(1)K &= Kx(2^{2n})h(2^n)K = Kh(2^n)K \text{ for } n > 0, \end{aligned}$$

and

$$Kx(1)h(2^n)x(1)K = \begin{cases} Kx(1)h(2^n)K & \text{if } n > 0, \\ Kh(2^n)x(1)K & \text{if } n < 0, \\ Kh(1)K & \text{if } n = 0. \end{cases}$$

Similarly, since $w(t)x(1)w(t)^{-1} = y(-t^2)$,

$$\begin{aligned} Kx(1)w(2^n)K &= Kw(2^n)y(-2^{-2n})K = Kw(2^n)K \text{ for } n < 0, \text{ and} \\ Kw(2^n)x(1)K &= Ky(-2^{-2n})w(2^n)K = Kw(2^n)K \text{ for } n < 0. \end{aligned}$$

Therefore, a complete set of double coset representatives for $K\backslash G/K$ is given by

$$\begin{array}{rcl} w(2^n) & : & n \in \mathbb{Z}, \\ h(2^n) & : & n \in \mathbb{Z}, \\ w(2^n)x(1) & : & n \geq 0, \\ x(1)w(2^n) & : & n \geq 0, \\ x(1)w(2^n)x(1) & : & n \geq 0, \\ x(1)h(2^n) & : & n \geq 0, \\ h(2^n)x(1) & : & n < 0. \end{array}$$

Lemma 2.7 *The support of $\mathcal{H}$ is contained in $KW^{\text{aff}}K$.*

Proof: Let f be an element of $\mathcal{H}$. Recalling two of the Steinberg relations, $w(a)x(t) = y(-a^{-2}t)w(a)$ and $h(a)y(t) = y(a^{-2}t)h(a)$, and Proposition 2.6, if $n \geq 0$, then

$$\begin{aligned} f\big(w(2^n)x(1)\big) &= \chi\big(x(2^{2n+1})\big)f\big(w(2^n)x(1)\big) \\ &= f\big(x(2^{2n+1})w(2^n)x(1)\big) \\ &= f\big(w(2^n)y(2)x(1)\big) \\ &= f\big(w(2^n)x(1)\big)\chi\big(x(-1)y(2)x(1)\big) \\ &= if\big(w(2^n)x(1)\big), \end{aligned}$$

$$\begin{aligned} f\big(x(1)w(2^n)\big) &= f\big(x(1)w(2^n)x(-2^{2n+1})\big) \\ &= \chi\big(x(1)y(2)x(-1)\big)f\big(x(1)w(2^n)\big) \\ &= if\big(x(1)w(2^n)\big), \end{aligned}$$

$$\begin{aligned} f\big(x(1)w(2^n)x(1)\big) &= f\big(x(1)w(2^n)x(1)x(-2^{2n+1})\big) \\ &= \chi\big(x(1)y(2)x(-1)\big)f\big(x(1)w(2^n)x(1)\big) \\ &= if\big(x(1)w(2^n)x(1)\big), \end{aligned}$$

$$\begin{aligned} f\big(x(1)h(2^n)\big) &= f\big(x(1)h(2^n)y(2^{2n+1})\big) \\ &= \chi\big(x(1)y(2)x(-1)\big)f\big(x(1)h(2^n)\big) \\ &= if\big(x(1)(2^n)\big), \end{aligned}$$

and, if $n < 0$, then

$$\begin{aligned} f\big(h(2^n)x(1)\big) &= f\big(y(2^{-2n+1})h(2^n)x(1)\big) \\ &= f\big(h(2^n)x(1)\big)\chi\big(x(-1)y(2)x(1)\big) \\ &= if\big(h(2^n)x(1)\big). \end{aligned}$$

Therefore, $f = 0$ on all double cosets besides $Kh(2^n)K$ and $Kw(2^n)K$. ■

In order to study the structure of $\mathcal{H}$, two subalgebras corresponding to the representatives w_0 and w_1 of the two generators of the affine Weyl group are introduced. For $i = 0, 1$, define

1. P_i to be the group generated by K and w_i,

2. V_i to be the subspace $P_i\phi_0$ of V,

3. U_i to be the induced representation $\mathrm{ind}_K^{P_i}\chi$,

4. $\mathcal{H}_i$ to be the Hecke subalgebra $\mathcal{H}(P_i//K;\chi)$.

Lemma 2.8 *V_0 is 2-dimensional and V_1 is 1-dimensional.*

Proof: It was already seen that the space $\mathbb{C}\phi_0$ is K-stable; in addition, $w_1\phi_0 \in \mathbb{C}\phi_0$, so

$$V_1 = P_1\phi_0 = \mathbb{C}\phi_0.$$

It will now be shown that $V_0 = \mathbb{C}\phi_0 \oplus \mathbb{C}\phi_1$. The element $w_0 = w(1/2)$ acts on ϕ_0 and ϕ_1 by

$$\begin{aligned} w(1/2)\phi_0(y) &\in \mathbb{C}\widehat{\phi}_0(y/2) = \mathbb{C}\phi_0(y/2) = \mathbb{C}\phi_1(y), \\ w(1/2)\phi_1(y) &\in \mathbb{C}\widehat{\phi}_1(y/2) = \mathbb{C}\phi_{-1}(y/2) = \mathbb{C}\phi_0(y). \end{aligned}$$

Since $y(2) = w_1x(-2)w_1^{-1}$ and $x(-2)\phi_{-1} \in \mathbb{C}\phi_0 \oplus \mathbb{C}\phi_{-1}$, one has

$$y(2)\phi_1 = w_1x(-2)w_1^{-1}\phi_1 = \frac{1}{2}\gamma_1^{-1}w_1x(-2)\phi_{-1} \in \mathbb{C}\phi_0 \oplus \mathbb{C}\phi_1.$$

Hence, $K\phi_0$, $w_0\phi_0$, $K\phi_1$, and $w_0\phi_1$ are all in $\mathbb{C}\phi_0 \oplus \mathbb{C}\phi_1$. ■

Lemma 2.9 *U_0 is 6-dimensional and U_1 is 2-dimensional.*

Proof: The dimension of U_i is equal to the index of K in P_i. In the $i = 1$ case, the element w_1 normalizes K, giving that $P_1 = K \cup Kw_1$, so $\dim U_1 = 2$. For the $i = 0$ case, the group P_0 is exactly the group K_{α_0} from Section 1.4. Under the map e_{α_0}, the linear subgroup $\underline{P}_0$ of $\underline{G}$ is isomorphic to $\mathrm{SL}_2(\mathbb{Z}_2)$ and $\underline{K}$ is isomorphic to the group

$$\left\{ \begin{bmatrix} a & b \\ 4c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}_2) : a, b, c, d \in \mathbb{Z}_2 \right\}.$$

Therefore, the index of K in P_0 is equal to the index of the first congruence subgroup $\Gamma_0(4)$ in $\mathrm{SL}_2(\mathbb{Z})$. This, in turn, is equal to the index of the subgroup of upper triangular matrices in $\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$. Since $\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$ has 48 elements and 8 of them are upper triangular, the dimension of U_0 is equal to 6. ■

Lemma 2.10 *Both $\mathcal{H}_0$ and $\mathcal{H}_1$ are 2-dimensional.*

Proof: The group P_i is contained in the parahoric subgroup $I \cup Iw_iI$, which intersects W^{aff} at 1 and w_i. Therefore, by Lemma 2.7, any element of $\mathcal{H}_i$ can only be supported on K and Kw_iK, so $\mathcal{H}_i$ is at most 2-dimensional. On the other hand, the characteristic function of K is the identity element of $\mathcal{H}$, and hence of $\mathcal{H}_i$, so $\mathcal{H}_i$ is at least 1-dimensional. By Proposition 1.6,

$$\mathrm{Hom}_{P_i}(V_i^*, U_i) = \mathrm{Hom}_K(V_i^*, \chi) = \mathrm{Hom}_K(\bar{\chi}, V_i) = V_i^{K,\bar{\chi}} = \mathbb{C}\phi_0,$$

and so there exists a P_i-homomorphism from V_i^* to U_i. For both $i = 0$ and $i = 1$, the dimension of V_i is less than the dimension of U_i, hence U_i must be reducible. Therefore, $\mathrm{End}(U_i)$, which by Proposition 1.8 is isomorphic to $\mathcal{H}_i$, cannot be 1-dimensional. ■

Lemma 2.11 *There exist elements T_0 of $\mathcal{H}_0$ and T_1 of $\mathcal{H}_1$ which satisfy the quadratic relations*

$$(T_0 - 2)(T_0 + 1) = 0 \qquad \textit{and} \qquad (T_1 - 1)(T_1 + 1) = 0.$$

Proof: Let T_i be the element of $\mathcal{H}_i$ supported on Kw_iK normalized as follows. Since $\mathcal{H}_i$ is 2-dimensional, T_i acts as an endomorphism on U_i with two eigenvalues λ_i and μ_i corresponding to eigenspaces of dimensions m_i and n_i. By Proposition 1.8, since T_i is not supported on K, T_i acts as a trace-zero endomorphism on U_i, giving that

$$\lambda_i m_i + \mu_i n_i = 0.$$

By Frobenius reciprocity, V_i is one of the eigenspaces, say that of dimension m_i. The element T_i is normalized to act by $\mu_i = -1$ on the eigenspace of dimension n_i (the one not containing ϕ_0). The other eigenvalue then is given by

$$\lambda_i = \frac{n_i}{m_i} = \begin{cases} 4/2 = 2 & \text{if } i = 0, \\ 1/1 = 1 & \text{if } i = 1. \end{cases}$$

■

Lemma 2.12 *The support of $\mathcal{H}$ is $KW^{\text{aff}}K$.*

Proof: From Lemma 2.7, it is enough to construct nonzero elements of $\mathcal{H}$ supported on each K-double coset of W^{aff}.

Let w be a representative of an element s of W^{aff}, and write a minimal expression for s, say as the product $s = s_{i_1} \ldots s_{i_m}$ of simple reflections s_0 and s_1. Define the element T_w of $\mathcal{H}$ by $T_w = T_{i_1} \cdots T_{i_m}$. The minimal expression for s is unique since there are no braid relations in W^{aff}, and hence, T_w is well-defined. The Hecke algebra $\mathcal{H}$ has a one-dimensional representation $\mathbb{C}\phi_0$ on which T_0 and T_1 act by the scalars 2 and 1, respectively. In particular, T_w acts on ϕ_0 as a product of 2's and 1's, so $T_w \neq 0$. Since w_0 and w_1 are representatives of s_0 and s_1 in G, the support of T_w satisfies

$$\operatorname{supp}(T_w) = Kw_{i_1}Kw_{i_2}K \cdots Kw_{i_m}K \subset Iw_{i_1}Iw_{i_2}I \cdots Iw_{i_m}I = IwI,$$

the last equality holding since the expression of s in terms of simple reflections is minimal. The double coset IwI is the union of finitely many K-double cosets; however, elements of $\mathcal{H}$ are not supported outside of those K-double cosets parametrized by W^{aff}. Therefore, the support of T_w must be equal to KwK. The set of nonzero elements $\{T_w : w \in W^{\text{aff}}\}$ forms a basis for $\mathcal{H}$. ■

Theorem 2.13 *Abstractly, $\mathcal{H}$ is the algebra generated by T_0 and T_1 subject to the quadratic relations $(T_0 - 2)(T_0 + 1) = 0$ and $(T_1 - 1)(T_1 + 1) = 0$. In particular, $\mathcal{H}$ is isomorphic to the Iwahori-Hecke algebra of* $\mathrm{PGL}_2(\mathbb{Q}_2)$.

Proof: Let A be the Iwahori-Hecke algebra of $\mathrm{PGL}_2(\mathbb{Q}_2)$. Then, A is generated by two elements t_0, t_1 subject only to the quadratic relations

$$(t_0 - 2)(t_0 + 1) = 0 \qquad \text{and} \qquad (t_1 - 1)(t_1 + 1) = 0.$$

For w a representative of s in W^{aff}, write $s = s_{i_1} \ldots s_{i_m}$ as a reduced expression, and define the element t_w of A by $t_w = t_{i_1} \ldots t_{i_m}$. These elements t_w form a basis for A (see [5]). Define a map $A \to \mathcal{H}$ by $t_w \mapsto T_w$. Since $\mathcal{H}$ satisfies the same relations as A, it is a homomorphism that sends a basis to a basis and is thus an isomorphism. ■

CHAPTER 3

WEIL REPRESENTATION OF $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{Q}_2)$

Throughout this chapter, $\mathcal{W}$ is a symplectic vector space over $\mathbb{Q}_2$ of dimension $2n$, $\mathcal{X} + \mathcal{Y}$ is a complete polarization of $\mathcal{W}$, $\underline{G} = \mathrm{Sp}(\mathcal{W})$ is the symplectic group, $G = \widetilde{\mathrm{Sp}}(\mathcal{W})$ is a central extension of $\underline{G}$ by $\mathbb{C}^\times$, $\psi = \psi_{1/2}$ is an additive character of $\mathbb{Q}_2$ of conductor $c = 1$, and $V = S(\mathcal{Y})$ is the Schrödinger model of the Weil representation of G with respect to ψ. Recall that the elements $x(b)$, $h(b)$, and w in G are the lifts of

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} b & 0 \\ 0 & {}^{\intercal}b^{-1} \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{respectively,}$$

and that the Weil representation (ω, V) is given by

$$\begin{aligned} x(b)\phi(y) &= \psi({}^{\intercal}yby)\phi(y) \\ h(b)\phi(y) &= \beta_b|\det(b)|^{1/2}\phi({}^{\intercal}by) \\ w\phi(y) &= \gamma_1\widehat{\phi}(y). \end{aligned}$$

Since $x(b)$ can be expressed as a product of elements in the positive root groups, the usual constant α_b is the product of some of the α_t from the $\widetilde{\mathrm{SL}}_2$ case, each of which is equal to 1. Similarly, if b is diagonal, then $h(b)$ is the product of some of the $h_\alpha(t)$, so the constant β_b is a 4th root of unity. In particular, the central extension $G = \widetilde{\mathrm{Sp}}(\mathcal{W})$ could be taken to be a two-fold central extension.

As a central extension of a Chevalley group, G is generated by $x_\alpha(t)$ for $\alpha \in \Phi$ and $t \in \mathbb{Q}_2$, and is subject to Steinberg's relations for covering groups (see Section 1.4.1). Using the notation of Section 1.3, for a positive root α, X_α is one of

$$\begin{bmatrix} E_{ji} & 0 \\ 0 & -E_{ij} \end{bmatrix}, \quad \begin{bmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 0 & E_{ii} \\ 0 & 0 \end{bmatrix}.$$

In the first case, $x_\alpha(t) = h(b)$ with $b = 1 + tX_\alpha$; in the second and third case, $x_\alpha(t) = x(b)$ with $b = tX_\alpha$. The root α is a short root in the first two cases and a long root in the third case.

Define the elements $w_0, w_1, \ldots, w_n$ of G using the same formulas as in the linear case (see Section 1.4). These elements are representatives in G of the simple generators $\{s_0, s_1, \ldots, s_n\}$ of the affine Weyl group W^{aff}.

3.1 A Minimal Type for $p = 2$

Let I be the Iwahori subgroup of G; that is, I is the full inverse image of the Iwahori subgroup $\underline{I}$ of $\underline{G}$, so that I is generated by

$$\left\{h_\alpha(t) : \alpha \in \Phi, t \in \mathbb{Z}_2^\times\right\} \cup \left\{x_\alpha(t) : \begin{array}{l} t \in \mathbb{Z}_2 \text{ for } \alpha > 0 \\ t \in 2\mathbb{Z}_2 \text{ for } \alpha < 0 \end{array}\right\}.$$

For α a positive long root, the action of $x_\alpha(t)$ (on its particular coordinate) mirrors that of the action of $x(t)$ in the $\widetilde{\mathrm{SL}}_2$ setting. By looking at each of the long roots, one sees that the Weil representation can only have Iwahori-fixed vectors if $p \neq 2$, in which case the fixed vector must be a multiple of the characteristic function of the standard lattice $\mathbb{Z}_p^n$. This, along with the preceding chapter, motivates the definition (in the $p = 2$ setting) of the subgroup K of I generated by

$$\left\{h_\alpha(t) : \alpha \in \Phi, t \in \mathbb{Z}_2^\times\right\} \cup \left\{x_\alpha(t) : \begin{array}{l} t \in \mathbb{Z}_2 \text{ for } \alpha > 0, \text{ short} \\ t \in 2\mathbb{Z}_2 \text{ for } \alpha > 0, \text{ long} \\ t \in 2\mathbb{Z}_2 \text{ for } \alpha < 0 \end{array}\right\}.$$

The group K has index 2^n in I.

A very nice way to think of K is the following. Recall the general definition of the Iwahori subgroup of $\mathrm{Sp}_{2n}(\mathbb{Z}_2)$ as the inverse image (under projection modulo 2) of the Borel subgroup B of the finite symplectic group $\mathrm{Sp}_{2n}(2)$. The finite orthogonal group $\mathrm{O}_{2n}(2)$ is defined as the set of linear operators of a $2n$-dimensional vector space over $\mathbb{F}_2$ under which a symmetric quadratic form q is invariant. Associated with q is the bilinear form (x, y) given by $(x, y) = q(x + y) + q(x) + q(y)$, which is clearly symmetric. However, in characteristic 2, a symmetric form is also skew-symmetric, so $\mathrm{O}_{2n}(2)$ is a subgroup of $\mathrm{Sp}_{2n}(2)$. Let B' be the Borel subgroup of the finite orthogonal group $\mathrm{O}_{2n}(2)$, realized as a subgroup of B. As Chevalley groups, B' is precisely the subgroup of B which is generated by the short root groups (see [1]). Therefore, under this identification, $\underline{K}$ is the pull-back of B' to $\mathrm{Sp}_{2n}(\mathbb{Z}_2)$ and K is the full inverse image of $\underline{K}$ in the central extension G.

$$\begin{tikzcd} \underline{K} \arrow[r] \arrow[d] & \underline{I} \arrow[r] \arrow[d] & \mathrm{Sp}_{2n}(\mathbb{Z}_2) \arrow[d] \\ B' \arrow[r] & B \arrow[r] & \mathrm{Sp}_{2n}(2) \end{tikzcd}$$

Lemma 3.1 *Let $\phi_0 \in V$ be the characteristic function of the standard lattice $\mathbb{Z}_2^n$. Then, $\mathbb{C}\phi_0$ is a K-stable subspace of V.*

Proof: Consider the action of the generators of K on ϕ_0, letting $y = (y_1, \ldots, y_n)$ be an element of $\mathbb{Z}_2^n$.

1. A generator $h_\alpha(t)$ is of the form $h(b)$ for b, a diagonal matrix with entries in $\mathbb{Z}_2^\times$. Then, ${}^\intercal by$ is in $\mathbb{Z}_2^n$, so $h_\alpha(t)\phi_0 \in \mathbb{C}\phi_0$.
2. Let $t \in \mathbb{Z}_2$ and α a short root so that $x_\alpha(t)$ is of the form $h(b)$ with $b = 1 + tE_{ij}$. Then, ${}^\intercal b$ acts on y by replacing y_j with $y_j + ty_i$, which is still in $\mathbb{Z}_2^n$. Therefore, $x_\alpha(t)\phi_0 \in \mathbb{C}\phi(0)$.
3. Let $t \in \mathbb{Z}_2$ and α a short root so that $x_\alpha(t)$ is of the form $x(b)$ with $b = t(E_{ij} + E_{ji})$. Then, ${}^\intercal yby = 2ty_iy_j \in 2\mathbb{Z}_2$, and hence, $x_\alpha(t)\phi_0 = \phi_0$.
4. Let $t \in 2\mathbb{Z}_2$ and α a long root. Then, $x_\alpha(t)$ is of the form $x(b)$ with $b = tE_{ii}$, and ${}^\intercal yby = ty_i^2 \in 2\mathbb{Z}_2$. Therefore, $x_\alpha(t)\phi_0 = \phi_0$.

Therefore, $\mathbb{C}\phi_0$ is preserved by each generator of K and the lemma is proved. ■

Proposition 3.2 *Let $\bar{\chi}$ be the character of K defined by $\omega(k)\phi_0 = \bar{\chi}(k)\phi_0$, which acts on the nontrivial subspace $V^{K,\bar{\chi}} = \{\phi \in V \,:\, \omega(k)\phi = \bar{\chi}(k)\phi$ for all $k \in K\}$. Then*

1. *$\bar{\chi}\big(x_\alpha(t)\big) = 1$, for α a positive long root and $t \in 2\mathbb{Z}_2$.*
2. *$\bar{\chi}\big(x_\alpha(-1)x_{-\alpha}(t)x_\alpha(1)\big) = \psi(-t/4)$, for α a positive long root and $t \in 2\mathbb{Z}_2$.*
3. *$V^{K,\chi} = \mathbb{C}\phi_0$.*

Proof: The first part follows from the proof of the previous lemma. Since the action of $x_\alpha(-1)x_{-\alpha}(t)x_\alpha(1)$ on ϕ_0 is the same as the $\widetilde{\mathrm{SL}}_2$ action of $x(-1)y(t)x(1)$, the second part is a consequence of Lemma 2.6. For the third part, it suffices to show that $V^{K,\chi} \subset \mathbb{C}\phi_0$. Consider a function ϕ in $V^{K,\chi}$, which is the tensor product of functions on $\mathbb{Q}_2$. By Lemma 2.6, each piece of ϕ must be a multiple of the characteristic function on $\mathbb{Z}_2$. Hence, the tensor product ϕ must be a multiple of the characteristic function on the standard lattice; that is, ϕ must be an element of $\mathbb{C}\phi_0$. ■

3.2 The Associated Hecke Algebra

The Hecke algebra $\mathcal{H} = \mathcal{H}(G//K;\chi)$ for this even type of the Weil representation is

$$\mathcal{H} = \{f \in C_c^\infty \,:\, f(k_1xk_2) = \chi(k_1)f(x)\chi(k_2) \text{ for all } k_i \in K, x \in G\}.$$

Lemma 3.3 *The support of $\mathcal{H}$ is contained in $KW^{\mathrm{aff}}K$.*

Proof: As before, an element f of $\mathcal{H}$ is determined by its value on a set of representatives of $K\backslash G/K$ or, equivalently, of $\underline{K}\backslash\underline{G}/\underline{K}$. By [5], the $\underline{I}$-double cosets are parametrized by W^{aff}. In addition, the I/K- and $K\backslash I$-cosets have as representatives products of elements of the form $x_\alpha(1)$ for α positive and long. Therefore, a typical double coset of $K\backslash G/K$ is of the form Kx_1wx_2K where w is a representative of an element of the affine Weyl group and x_1, x_2 are products of some of those $x_\alpha(1)$'s. Recall that, for any two long roots α, β with $\beta \neq -\alpha$, the root groups $\mathfrak{X}_\alpha$ and $\mathfrak{X}_\beta$ commute with each other.

Suppose that $x_\alpha(1)$ occurs in x_1 and that $w^{-1}x_\alpha(1)w = x_\beta(t)$ for some root β and some $t \in 2\mathbb{Z}_2$. It may be assumed that $x_1 = x_1'x_\alpha(1)$. If $x_{-\beta}(1)$ does not occur in x_2, then

$$Kx_1'x_\alpha(1)wx_2K = Kx_1'wx_2x_\beta(t)K = Kx_1'wx_2K.$$

If $x_{-\beta}(1)$ does occur in x_2, say $x_2 = x_2'x_{-\beta}(1)$, then

$$\begin{aligned} Kx_1'x_\alpha(1)wx_2K &= Kx_1'wx_2'x_\beta(t)x_{-\beta}(1)K \\ &= Kx_1'wx_2'x_{-\beta}(1)\big(x_{-\beta}(-1)x_\beta(t)x_{-\beta}(1)\big)K \\ &= Kx_1'wx_2K, \end{aligned}$$

with the last equality following from the fact that the subgroup of $\mathrm{SL}_2(\mathbb{Z}_2)$ consisting of matrices of the form $\begin{bmatrix} a & 2b \\ 2c & d \end{bmatrix}$ is a normal subgroup of the Iwahori subgroup of $\mathrm{SL}_2(\mathbb{Z}_2)$. In either case, if $t \in 2\mathbb{Z}_2$, then $x_\alpha(1)$ may be moved across and absorbed into K.

Supposing now that $x_\alpha(1)$ occurs in x_1 with $w^{-1}x_\alpha(1)w = x_\beta(t)$ for $t \notin 2\mathbb{Z}_2$, suppose further that $x_\beta(1)$ occurs in x_2. Since $wx_\beta(1)w^{-1} = x_\alpha(1/t)$, a similar argument to that above permits the assumption that $1/t \notin 2\mathbb{Z}_2$; that is, it may be assumed that $t \in \mathbb{Z}_2^\times$, or that $1 + t \in 2\mathbb{Z}_2$. Writing $x_1 = x_1'x_\alpha(1)$ and $x_2 = x_2'x_\beta(1)$, one has

$$Kx_1'x_\alpha(1)wx_2'x_\beta(1)K = Kx_1'wx_2'x_\beta(t+1)K = Kx_1'wx_2'K.$$

Therefore, it may be assumed that such an $x_\alpha(1)$ occurring in x_1 precludes the appearance of the corresponding $x_\beta(1)$ in x_2.

In summary, if $x_\alpha(1)$ occurs in x_1 with $w^{-1}x_\alpha(1)w = x_\beta(t)$, it may be assumed that $t \notin 2\mathbb{Z}_2$ and that $x_\beta(1)$ does not occur in x_2. For $f \in \mathcal{H}$, writing $x_1 = x_\alpha(1)x_1'$, one has

$$
\begin{aligned}
f(x_1 w x_2) &= f(x_1 w x_2)\chi\big(x_{-\beta}(2/t)\big) \\
&= f\big(x_1 w x_2 x_{-\beta}(2/t)\big) \\
&= f\big(x_1 w x_{-\beta}(2/t) x_2\big) \\
&= f\big(x_1 x_{-\alpha}(2) w x_2\big) \\
&= f\big(x_\alpha(1) x_{-\alpha}(2) x_\alpha(-1) x_1 w x_2\big) \\
&= \chi\big(x_\alpha(1) x_{-\alpha}(2) x_\alpha(-1)\big) f(x_1 w x_2) \\
&= i f(x_1 w x_2),
\end{aligned}
$$

giving that $f(x_1 w x_2) = 0$. Therefore, $\mathcal{H}$ is not supported on any coset $Kx_1 w x_2 K$ when x_1 is nontrivial. A symmetrical calculation shows that $\mathcal{H}$ is not supported on any coset Kwx_2K when x_2 is nontrivial. Hence, the support of $\mathcal{H}$ is at most the set of double cosets parametrized by the affine Weyl group. ■

As in the $\widetilde{\mathrm{SL}}_2$ setting, in order to study the structure of $\mathcal{H}$, subalgebras $\mathcal{H}_0, \ldots, \mathcal{H}_n$ are introduced. For $i = 0, \ldots, n$ define

1. P_i to be the group generated by K and w_i,
2. V_i to be the subspace $P_i \phi_0$ of V,
3. U_i to be the induced representation $\mathrm{ind}_K^{P_i} \chi$,
4. $\mathcal{H}_i$ to be the Hecke subalgebra $\mathcal{H}(P_i//K;\chi)$.

Lemma 3.4 *The dimension of V_i is*

$$
\begin{cases} 2 & \text{if } \; i = 0, \\ 1 & \text{if } \; i = 1, \ldots, n. \end{cases}
$$

Proof. The action of $w_0 = w_{\alpha_*}(1/2) = w_{2\lambda_1}(1/2)$ on the first component of $S(\mathcal{Y})$ is the same as the action of $w(1/2)$ in the $\widetilde{\mathrm{SL}}_2$ case. Similarly, the action of $w_n = w_{\alpha_n}(1) = w_{2\lambda_n}(1)$ on the last component of $S(\mathcal{Y})$ is the same as the action of $w(1)$ in the $\widetilde{\mathrm{SL}}_2$ case. Therefore, by Lemma 2.8, the dimension of V_0 is 2, and the dimension of V_n is 1. For $i = 1, \ldots, n-1$, w_i is of the form $h(b)$, hence P_i fixes $\mathbb{C}\phi_0$, and the dimension of V_i is 1. ■

Lemma 3.5 *The dimension of U_i is*

$$
\begin{cases} 6 & \text{if } \; i = 0, \\ 3 & \text{if } \; i = 1, \ldots, n-1, \\ 2 & \text{if } \; i = n. \end{cases}
$$

Proof: The dimension of U_i is equal to the index of K in P_i. In the $i = n$ case, the element w_n normalizes K, giving that $Kw_nK = K \cup Kw_n$, so the index of K in P_n is 2. In the other cases, using the notation of Section 1.4, the group P_i is exactly the group $K_{\alpha_i}K$, so the index of K in P_i is equal to the index of $\underline{K} \cap \underline{K}_{\alpha_i}$ in $\underline{K}_{\alpha_i}$. This index will be studied via the map $e_{\alpha_i} : \mathrm{SL}_2(\mathbb{Q}_2) \to \underline{G}$, under which $\underline{K}_{\alpha_i}$ is isomorphic to $\mathrm{SL}_2(\mathbb{Z}_2)$. For $i = 0$, the group $\underline{K} \cap \underline{K}_{\alpha_0}$ is generated by $\underline{x}_{\alpha_0}(t)$ and $\underline{x}_{-\alpha_0}(4t)$ for $t \in \mathbb{Z}_2$. Therefore, $\underline{K} \cap \underline{K}_{\alpha_0}$ is isomorphic to the first congruence subgroup of $\mathrm{SL}_2(\mathbb{Z}_2)$, which is a subgroup of index 6 (see the proof of Lemma 2.9). For $i = 1, \ldots, n-1$, the group $\underline{K} \cap \underline{K}_{\alpha_i}$ is generated by $\underline{x}_{\alpha_i}(t)$ and $\underline{x}_{-\alpha_i}(2t)$ for $t \in \mathbb{Z}_2$. Therefore, $\underline{K} \cap \underline{K}_{\alpha_i}$ is isomorphic to the Iwahori subgroup of $\mathrm{SL}_2(\mathbb{Z}_2)$, which is a subgroup of index 3. ■

Lemma 3.6 *Each $\mathcal{H}_i$ is 2-dimensional.*

Proof: The group P_i is contained in the parahoric subgroup $I \cup Iw_iI$, which intersects W^{aff} only at 1 and w_i. By Lemma 3.3, the only double cosets on which $\mathcal{H}_i$ is supported are K and Kw_iK, so $\mathcal{H}_i$ is at most 2-dimensional. The characteristic function on K is the identity element of each $\mathcal{H}_i$, so $\mathcal{H}_i$ is at least 1-dimensional. As in the proof of Lemma 2.10,

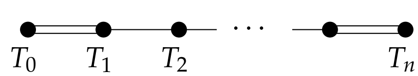
$$\mathrm{Hom}_{P_i}(V_i^*, U_i) = \mathrm{Hom}_K(V_i^*, \chi) = \mathrm{Hom}_K(\bar{\chi}, V_i) = V_i^{K,\bar{\chi}}.$$

Since $\dim V_i \leq \dim U_i$, the induced representation U_i must be reducible; hence, $\mathrm{End}(U_i) = \mathcal{H}_i$ cannot be 1-dimensional. ■

Lemma 3.7 *For $i = 0, \ldots, n$, there exists elements T_i of $\mathcal{H}$, supported on Kw_iK, which satisfy the quadratic relations*

$$\begin{cases} (T_i + 1)(T_i - 2) = 0 & \textit{for } i = 0, 1, \ldots, n-1 \\ (T_n + 1)(T_n - 1) = 0 \end{cases}$$

and the braid relations given by the following Coxeter diagram.



Proof: As $\mathcal{H}_i$ is 2-dimensional, an element with support Kw_iK acts as a trace-zero endomorphism of U_i by two eigenvalues λ_i and μ_i with corresponding eigenspaces of dimension m_i and n_i. Let V_i be the eigenspace of dimension m_i. Choose such an element T_i normalized to act by $\mu_i = -1$ on the other eigenspace (that which does not contain ϕ_0). Since

$$\lambda_i m_i + \mu_i n_i = 0,$$

T_i acts on V_i by

$$\lambda_i = \frac{n_i}{m_i} = \begin{cases} 4/2 = 2 & \text{if } i = 0, \\ 2/1 = 2 & \text{if } i = 1, \ldots, n-1, \\ 1/1 = 1 & \text{if } i = n, \end{cases}$$

which gives the desired quadratic relations.

Let $s_{i_1} s_{i_2} s_{i_1} \cdots = s_{i_2} s_{i_1} s_{i_2} \cdots$ be a braid relation in the affine Weyl group, and let w_{i_1}, w_{i_2} be the corresponding representatives of s_{i_1}, s_{i_2} in the group G. The support of $T_{i_1} T_{i_2} T_{i_1} \cdots$ satisfies

$$K w_{i_1} K w_{i_2} K w_{i_1} K \cdots \subset I w_{i_1} I w_{i_2} I w_{i_1} I \cdots = I w_{i_1} w_{i_2} w_{i_1} \cdots I,$$

while the support of $T_{i_2} T_{i_1} T_{i_2} \cdots$ satisfies

$$K w_{i_2} K w_{i_1} K w_{i_2} K \cdots \subset I w_{i_2} I w_{i_1} I w_{i_2} I \cdots = I w_{i_2} w_{i_1} w_{i_2} \cdots I.$$

The last equalities in these two statements follow from the fact that the corresponding expressions in the affine Weyl group are minimal expressions. In addition, the braid relation of W^{aff} implies that these two I-double cosets are equal. However, the elements of $\mathcal{H}$ are only supported on K-double cosets of W^{aff}, so the two functions $T_{i_1} T_{i_2} T_{i_1} \cdots$ and $T_{i_2} T_{i_1} T_{i_2} \cdots$ must be supported on

$$K w_{i_1} w_{i_2} w_{i_1} \cdots K = K w_{i_2} w_{i_1} w_{i_2} \cdots K.$$

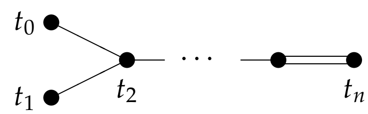
It follows that they must be equal up to multiplication by a scalar. By considering the representation $\mathbb{C}\phi_0$ of $\mathcal{H}$ on which T_{i_k} acts by either 1 or 2, one concludes that the functions must, in fact, be equal. Therefore, every braid relation of W^{aff} yields a corresponding braid relation of $\mathcal{H}$. ■

Lemma 3.8 *The support of $\mathcal{H}$ is $KW^{\mathrm{aff}}K$.*

Proof: From Lemma 3.3, it remains only to verify that $\mathcal{H}$ contains a nontrivial element supported on every K-double coset of W^{aff}. Let w be a representative of an element s of the affine Weyl group, and write a minimal expression $s = s_{i_1} \ldots s_{i_r}$ for s as the product of simple reflections. Define the element $T_w = T_{i_1} \ldots T_{i_r}$ of $\mathcal{H}$. The Hecke algebra satisfies the same braid relations as the affine Weyl group, so this definition of T_w is independent of the choice of minimal expression. As in the proof of the previous lemma, this element is supported on KwK. Since each T_{i_k} acts on ϕ_0 by 1 or 2, $T_w \neq 0$. Moreover, the set of nonzero elements $\{T_w : w \in W^{\mathrm{aff}}\}$ forms a basis for $\mathcal{H}$. ■

Theorem 3.9 *Abstractly, $\mathcal{H}$ is the algebra generated by $T_0, \ldots, T_n$ subject to the quadratic and braid relations from Lemma 3.7. In particular, $\mathcal{H}$ is isomorphic to the Iwahori-Hecke algebra of* $\mathrm{SO}_{2n+1}(\mathbb{Q}_2)$.

Proof: Let A be the Iwahori-Hecke algebra of $\mathrm{SO}_{2n+1}(\mathbb{Q}_2)$ with generators $t_0, \ldots, t_n$. Then, $(t_i - 2)(t_i + 1) = 0$ for all i and the braid relations satisfied by the t_i are given by the following Coxeter diagram.



For each $w \in W^{\mathrm{aff}}$, define an element $t_w = t_{i_1} \ldots t_{i_r}$ where $w = s_{i_1} \ldots s_{i_r}$ is a minimal expression for w. By [5], $\{t_w : w \in W^{\mathrm{aff}}\}$ forms a basis for A.

Let τ be the involution that exchanges the t_0 and t_1 vertices of the Coxeter diagram. The automorphism τ can replace t_0 as a generator of A (since $t_0 = \tau\, t_1 \tau$), which satisfies

$$\tau^2 = 1 \quad \text{and} \quad \tau\, t_1 \tau\, t_1 = t_0 t_1 = t_1 t_0 = t_1 \tau\, t_1 \tau.$$

Define a map on the generators from A to $\mathcal{H}$ by

$$\tau \mapsto T_n, \quad \text{and} \quad t_i \mapsto T_{n-i} \quad \text{for } i = 1, \ldots, n.$$

This map extends to a homomorphism which sends each basis element t_w of A to a basis element T_w of $\mathcal{H}$, so it must be an isomorphism. ■

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